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VECTORS AND VECTOR DIAGRAMS

APPLIED TO THE

ALTERNATING CURRENT CIRCUIT

VECTORS AND VECTOR DIAGRAMS

APPLIED TO THE ALTERNATING CURRENT CIRCUIT

WITH EXAMPLES OF THEIR USE IN THE THEORY OF TRANSFORMERS, AND OF SINGLE AND POLYPHASE MOTORS, ETC.

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WITH DIAGRAMS

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LIST OF CONVENTIONS, ABBREVIATIONS AND SYMBOLS EMPLOYED

CONVENTIONS AND ABBREVIATIONS

Arrow-heads in Diagrams—

Thin barb indicates vector of voltage.

Thick triangular barb indicates vector of current.

Double barb indicates vector of flux.

Capital Letters indicate vectors or complex numbers (*e.g.*, C the vector of current; Z the impedance of a circuit expressed as a complex number).

(For special convention regarding use of capital letters employed in Chapters IX. and X., see p. 186.)

Small Letters represent numerical values of vector quantities or other simple numbers (*e.g.*, *c* the number of amperes of current; *z* the number of ohms in the impedance of a circuit).

Rotation of Vectors is assumed to be clockwise.

Angles are considered positive when measured in a counter-clockwise sense.

j rotates a vector through 90° in a counter-clockwise sense.

Unit Vector is indicated by a single dash (′) when parallel to the axis of reference, and by double dashes (″) when normal to this axis.

Fluxes are expressed in maximum values, unless otherwise indicated.

Currents and Voltages are expressed in virtual (R.M.S.) values unless otherwise indicated.

Units.—The practical system of units is used throughout, except that fluxes are measured in C.G.S. units.

Abbreviations—

E.M.F., electromotive force.

R.M.S., root of mean squares (or virtual) value of an alternating quantity.

~, frequency in cycles per sec.

LIST OF SYMBOLS

ROMAN LETTERS

In most cases of vectors and complex numbers enumerated in the following list, the capital letter only is given. The corresponding small letter must consequently be taken as the numerical value of the quantity represented by the capital letter.

Reference is made to the page where the symbol is explained, or first used, when such a reference is likely to be of assistance.

		PAGE
b	Susceptance of circuit $\left(=\frac{x}{r^2+x^2}\right)$	60
C	Current (amperes).	
C_0	Magnetizing current, including iron loss current ($= C_i + C_m$).	
C_1	Primary current (of transformer, etc.).	
C_2	Secondary current (of transformer, etc.).	
C_i	Iron-loss current.	
C_m	Magnetizing current (wattless current).	
C_s	Current in primary circuit producing ampere-turns equal and opposite to C_2 $\left(C_s = -\frac{1}{k}C_2\right)$.	
C_x	Short-circuit current, or current with locked rotor.	
E	Voltage, total applied voltage.	
E_1	Primary applied voltage overcoming induced voltage in primary winding.	
E_2	Secondary induced voltage.	
E_a	Applied voltage overcoming voltage induced by flux, F_a	79
E_b	Applied voltage overcoming voltage induced by flux, F_b	80
E_m	Applied voltage overcoming voltage induced by total flux in iron-circuit	81
E_r	Applied voltage counterbalancing voltage induced by rotation	147
E_x	Voltage applied to winding to balance induced voltage due to leakage flux, F_1	78
F	Flux, total flux through primary winding.	
F_1	Primary leakage flux	74
F_2	Secondary leakage flux	91
F_a	Component of flux in iron path in phase with current	75
F_b	Component of flux in iron path perpendicular in phase to current	75
F_m	Total flux in iron path	74
g	Conductance of circuit $\left(=\frac{r}{r^2+x^2}\right)$	60
h	Flux per ampere-turn	41
h_1	Leakage flux per ampere-turn in primary	74
h_2	Leakage flux per ampere-turn in secondary	92
H_m	Flux in iron-path per ampere-turn	75
j	($=\sqrt{-1}$) operator producing rotation of a vector through 90° in counter-clockwise direction	42, 44

		PAGE
k	Ratio of transformation $= \frac{t_1}{t_2}$	105
K	Electrostatic capacity (in farads)	46
L	Coefficient of self-induction of primary winding due to total primary flux (Maxwell) in henrys	82
l, l_1	Coefficient of self-induction of primary winding due to primary leakage flux in henrys	83
l_2	Coefficient of self-induction of secondary winding due to secondary leakage flux in henrys.	
L'	Coefficient of self-induction of secondary winding due to total secondary flux in henrys	91
M	Coefficient of mutual induction between two coils due to total flux (Maxwell) in henrys	91
m_1	Flux per ampere-turn in iron path in phase with current $(= h_m \cos \alpha)$	75
m_2	Flux per ampere-turn in iron path perpendicular in phase to current $(= h_m \sin \alpha)$	75
n	Frequency of revolution	128
q	Ratio of $\frac{t_a}{t_2}$ in the compensated motor	162
r	Resistance.	
r_1	Resistance of primary winding.	
r_2	Resistance of secondary winding.	
r_m	$(= 2\pi m_2 t^2 \sim 10^{-8})$ a quantity having the dimensions of a resistance, which on multiplication by C_o^2 gives the power spent in iron-losses	80
s	Slip in cycles per sec. $(= \sim - n)$	124
t	Total number of turns of a winding.	
t_1	Total number of turns of primary winding.	
t_2	Total number of turns of secondary winding.	
t_a	Resolved component of stator winding along brush axis of repulsion motor	151
t_b	Resolved component of stator winding perpendicular to brush axis of repulsion motor	151
V	Voltage.	
x	Reactance $\left(= 2\pi \sim l = \frac{2\pi h t^2 \sim}{10^8}\right)$	78
	in circuit with capacity and inductance $\left(= 2\pi \sim l - \frac{1}{2\pi \sim K}\right)$	47
x_1	Reactance of primary winding due to leakage flux	102
x_2	Reactance of secondary winding due to leakage flux	102
x_m	$(= 2\pi m_1 t^2 \sim 10^{-8})$ a quantity having the dimensions of a reactance, which on multiplication by C_o^2 gives the wattless power of circuit	80
Y	Admittance $\left(= \frac{r}{r^2 + x^2} + j \frac{x}{r^2 + x^2}\right)$	58, 81
Z	Impedance $(= r - jx)$	54

		PAGE
Z_1	Impedance of primary winding due to primary resistance and leakage flux	104
Z_2	Impedance of secondary winding due to secondary resistance and leakage flux	104
Z_3	(In induction motor) impedance of rotor at any speed $= r_2 - j \frac{s}{\sim} x_2$	129
Z_4	(In induction motor) apparent impedance of rotor at any speed $= \frac{E_2}{C_2} = \left(\frac{\sim}{\sim - n} r_2 - jx_2\right)$	130
Z_5	(In single-phase induction motor) apparent impedance of rotor at any speed $= \frac{E_2}{C_2} = \left(\frac{\sim^2}{\sim^2 - n^2} r_2 - jx_2\right)$	139
Z_a	Impedance of armature and winding t_a in compensated motor	162
Z_m	Impedance of winding due to flux following iron path $= r_m - jx_m = (m_2 - jm_1)2\pi t^2 \sim 10^{-8}$	81

GREEK LETTERS

α	Angle of hysteretic advance	74
ϵ	Logarithmic base, 2·718.	
λ	Leakage factor (Hopkinson)	77
	Also in Chapter VII., angular displacement of brushes in repulsion motor	151
	And in Chapter VI., the angular displacement of a rotor coil from the polar axis	128, 129
λ_1	Leakage factor of primary winding	92
λ_2	Leakage factor of secondary winding	92
μ	Permeability.	
ν	Dispersion coefficient $(= - \lambda_1\lambda_2)$	95, 134
θ	Phase angle ($= 2\pi \sim t$, where t is time in secs.).	
σ	Dispersion coefficient $\left(1 - \frac{1}{\lambda_1\lambda_2}\right)$	95, 134
ϕ	Angle of phase difference between applied voltage and current of circuit.	

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INTRODUCTORY NOTE

VECTOR diagrams are now almost universally used to express the relationships existing in alternate-current circuits. The application of some system of vector algebra could hardly fail to follow this use of the diagrams, since these are usually neither convenient nor sufficiently accurate for actual calculations, although they are most valuable for exhibiting the general connection between the quantities in a circuit.

To Mr. C. P. Steinmetz belongs the credit of having first applied such a system of calculation to the whole range of problems which the electrical engineer is daily called upon to solve, and he has worked his system out in a thorough and masterly way. His notation is, however, so easy and flexible as to lead the unwary into the error of carrying the simplicity of its symbols into calculations where such application is no longer legitimate. Among difficulties which consequently arise none are more perplexing than those in connection with vector products. To obviate these, to keep clear throughout a problem the relative phases of the various vectors, and to discriminate between vectors and complex numbers, we have only found it necessary to alter the notation so as to distinguish between the unit vectors and the operator called j. It is the use of j for both unit vector and operator which seems to us to render the Steinmetz system confusing to the student, and the matter is not mended by the adoption of 1, as the unit vector along an axis perpendicular to that indicated by j. On the other hand, by retaining j, as operator only, we lose little of the flexibility of the notation of Steinmetz, yet we avoid the complications incidental to the more academic systems suggested by various mathematicians.

The whole subject of alternating-current phenomena bristles with difficulties; and it is not to be supposed that any student can make effective use of methods such as those developed in the following pages, till he has grasped the fundamental laws of the alternating-current circuit very thoroughly. For this reason we have assumed such knowledge on the part of the reader as an electrical engineering student in his third college year might fairly be expected to possess; and we think that, granting this knowledge, the methods outlined will be found of great practical use, especially as weapons of research.

In order that vector algebra may be easily applicable to general alternating-current problems, it is necessary to represent the waves of current, pressure, flux, etc., as sinusoidal in form; otherwise a simple rotating vector will not represent the alternating quantity, and the angle of phase difference is quite indefinite. This convention does not entail the assumption that the quantities mentioned do actually vary in accordance with a simple sine law, for this might be, in many cases, very far from the truth. It merely postulates in such cases the substitution of the "equivalent sine wave" for the complex general alternating wave.

Thus if $a_a, a_b \dots a_n$ represent successive instantaneous values of a general alternating-pressure wave, we shall designate it by e, where

$$e = \sqrt{\frac{a_a^2 + a_b^2 + a_c^2 + \dots a_n^2}{n}}$$

Now, a sine wave with maximum value $= \sqrt{2}e$ will have the same root-mean-square value, viz. e. Hence, such a wave may so far be considered as the equivalent of the general wave. Similarly, if a current wave have ordinates

$$b_a, b_b \dots b_n,$$

its root-mean-square (R.M.S.) value is

$$c = \sqrt{\frac{b_a^2 + b_b^2 \dots b_n^2}{n}}$$

Now, let the ordinate a_a correspond in time with the ordinate b_a. Then the mean power in the circuit having the above general current and pressure waves would be

$$P = \frac{a_a \,.\, b_a + a_b \,.\, b_b \,.\,.\,.\, a_n \,.\, b_n}{n}$$

But the mean power in a circuit having sine waves of pressure and current whose maxima were $\sqrt{2}e$ and $\sqrt{2}c$ respectively would be $ec \cos \phi$; where ϕ is the angular displacement between e and c.

Thus if the equivalent sine waves are to represent fully the general waves, not only must e and c be the R.M.S. values of the corresponding general waves, but also $ec \cos \phi$ must have the same value as "P," that is

$$ec \cos \phi = \frac{a_a \,.\, b_a + a_b \,.\, b_b + \,.\,.\,.\, a_n b_n}{n}$$

Or substituting the values for e and c above

$$\cos \phi = \frac{a_a \,.\, b_a + a_b \,.\, b_b + \,.\,.\,.\, a_n b_n}{\sqrt{a_a^2 + a_b^2 + \,.\,.\,.\, a_n^2} \,.\, \sqrt{b_a^2 + b_b^2 + \,.\,.\,.\, b_n^2}}$$

If these definitions of e and c are adhered to, then it is possible to treat almost all problems connected with alternating-current circuits as if the alternating waves were sine waves, and the employment of vector diagrams and vector algebra becomes generally justifiable.

Some word of explanation is also necessary as to the scope and arrangement of the chapters which follow. So far as possible each chapter has been arranged to cover a particular subject. This has the undesirable effect of making some of them rather long, but it also has the advantage of enabling any one who has once grasped the notation to make use of any special chapter without necessarily reading those in which for the moment he has no interest. The idea throughout is to link up vector diagrams with the corresponding algebra, and in order to do this it is essential that there should be a clear and definite understanding as to the meaning of terms which are in common use. It is in the hope of clearing away some of the difficulties arising from the variety of treatment which is to be met with, that a whole chapter has been devoted to the relationships of the various coefficients of self-induction, and of the leakage and dispersion factors, adopted by different schools.

sequence of this, any two vectors having the same
gth, inclination, and sense have exactly the same meaning,
represent either similar or identical vector quantities.

Rotors.—In some problems, vector quantities are repre-
ed by lines called "rotors," which have the same general
acteristics as the vectors described above, but which are
l as regards their position, or line of action.

or instance, we may represent a number of forces acting
solid body by lines which will be rotors, and not vectors,
the exact position at which they are applied to the
is a matter of importance. Their line of action is,
quently, definite and fixed.

rotor may thus be defined as a *localized vector*.

presentation of Alternating Quantities.—Vectors can
ployed for representing alternating quantities, as well
ntities having fixed values and directions. It is in this
tion that we are more particularly concerned with them.
ere are several different ways in which a vector may
ent an alternating quantity, depending on the particular
teristics of it with which we happen to be dealing.
ample, we may represent the flux in the air-gap of an
tor by a vector. The vector may be required specially
the direction of the flux with reference to the surface
armature, or it may be made to show the relative
of the flux and of the armature-induced voltage; or,
he vector might be used to indicate the successive
neous values of the flux at certain intervals of time.
e shall find that, when used to represent an alter-
urrent or voltage, the meaning to be assigned to the
on and length of the vector may vary, and, con-
y, must be definitely stated beforehand.

alternating quantities with which we have to deal
harmonic variation, passing through a series of
and negative values which recur successively, and
ne order, with the same time-interval.

y be well to consider the nature of this harmonic
in order to introduce the methods to be adopted
resentation.

Harmonic Variation.—Let OP be a line of constant length, which rotates in such a manner that the point O remains fixed, while P moves in the plane of the paper in a circle round O. Let O′P′ be the projection of OP on any arbitrarily chosen axis XX′; O′P′ is obtained by drawing perpendiculars OO′, PP′ from O and P to this axis. Then the projection O′P′ undergoes simple harmonic variation as P moves with uniform velocity round the circle. If θ is the angle which OP makes with a line drawn perpendicular to XX′, we may write

$$O'P' = a \sin \theta$$

where a is the constant length OP. The quantity $a \sin \theta$ will undergo simple harmonic variation if the angle θ varies uniformly. Hence the variation of O′P′ is said to be *sinusoidal.*

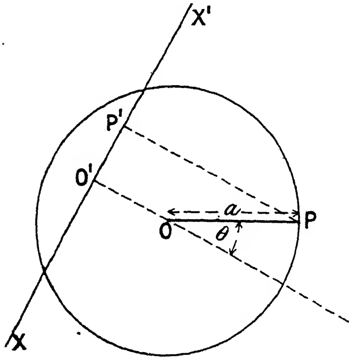

FIG. 2.—Simple harmonic variation.

If P were assumed to move round any closed figure other than a circle with centre O, the harmonic variation represented by the successive values of the projection of OP would no longer be "simple." It is, however, possible to analyze a variation of this character into a number of simple harmonic elements having various frequencies and amplitudes.

We shall assume in what follows that the alternating quantities with which we deal may be considered to undergo simple harmonic variation,[1] so that their values may be written in the form $a \sin \theta$, where a is the maximum value, and θ is an angle undergoing uniform change at the rate of $2\pi \sim$ radians per second, where $\sim$ represents the frequency of the quantities in cycles per second.

Polar Diagram.—The successive values of a quantity ich varies according to the simple harmonic law may be wn in another way.

---

[1] *Vide* Introductory Note.

Retaining the idea of a uniformly rotating line OP, let us w a diameter AB to the circle described by P, making parallel to the axis XX′ in Fig. 2. If we now describe circles on this diameter, making the diameter of each le equal to the radius OP (see Fig. 3), it can be shown the length OM, *i.e.* the ion of the rotating line OP off by the circles for any tion of OP, is equal to the ection OP′ of OP on the AB, and this again is the length as O′P′ cut off on arallel axis XX′ in Fig. 2.

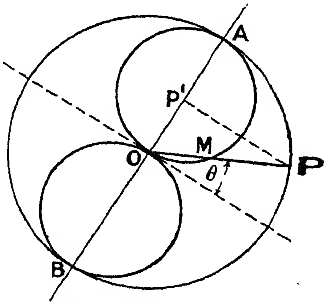

FIG. 3.—Polar diagram.

onversely, the circles may rawn with diameters on and the successive values variable quantity may be ied by imagining the axis AB to rotate in the opposite ion to that previously assumed for OP. The instan- is values are then the portions of the rotating axis cut the circles.

e small circles shown in Fig. 3 are called polar circles, ie diagram in which the changes in the alternating ty are obtained by means of these circles is called a *liagram.*

ating Vectors.—The rotating line OP in Figs. 2 and 3 called a Rotating Vector, and the variable length OM) or O′P′ (Fig. 2) may be made to represent the of any vector quantity which obeys the law of simple ic variation.

dently the two methods of deriving the successive aneous values of the quantity from a rotating vector, n in Figs. 2 and 3, are the same in principle. In either method may be employed in any case.

le Harmonic Function referred to Rectangular Co- s.—We have seen that the successive instantaneous
]

values of an alternating quantity, *e.g.* an alternating current, may be derived in the following ways:—(1) By taking the projections on a fixed axis of a constant and uniformly rotating line; (2) From the polar diagram, by observing the length cut off from the rotating line by the polar circles; (3) By putting in a succession of values for θ in an expression of the form $a \sin \theta$, in which the angle θ is assumed to pass through all values from $\theta = 0$ to $\theta = 360°$.

The values obtained by any of these means may be shown in the form of a curve referred to rectangular co-ordinates, in which time or angle is plotted horizontally, and instantaneous value vertically. When plotted in this way, we have the most complete representation of the changes undergone by the quantity.

In taking the projections of the rotating line, it will generally be convenient to take the vertical axis as the axis of projections; the horizontal axis then becomes the one from which the inclination θ of the vector is to be measured.

Let it be required, for example, to trace the variations of an alternating current having a maximum value of 10 amps.

We begin by drawing a circle of radius equal to 10 units (Fig. 4). The radius OP of this circle is then the rotating vector for the current. The value of the current for any position of the vector is shown by the length of the vertical projection ON to the same scale of units as that originally chosen for the vector itself.

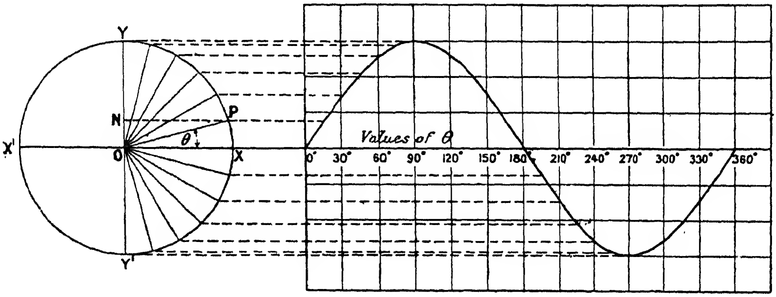

FIG. 4.—Harmonic variations plotted to rectangular co-ordinates.

By setting out horizontally a scale of angles (proportional to times), and transferring the vertical projections of OP corresponding to each value of the angle θ, we obtain the sinusoidal curve shown on the right of Fig. 4.

The same curve may be obtained by using the polar diagram (Fig. 5), the ordinates in this case being the lengths intercepted on the rotating line by the polar circle, instead of the vertical projections of the line.

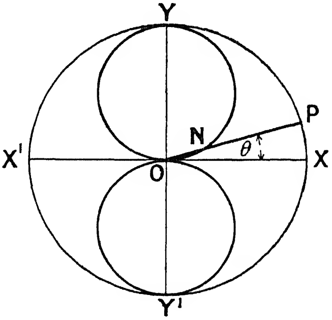

Fig. 5.—Polar diagram corresponding to Fig. 4.

In Fig. 4, the wave is shown plotted upon a base of degrees of angle, which also represents a scale of time, since degrees are only subdivisions of the complete period of an alternation of the current there represented.

The usual case occurring with alternating currents, voltages, etc., is this plotting of instantaneous values on a base which is proportional to time. It is evident, however, that the system of representation is perfectly general for all alternating vector quantities, whether their instantaneous values are functions of time or of some other factor.

For instance, in the case of a continuous-current generator, or motor, the field may often be assumed to be uniformly distributed as regards a particular axis. Thus, Fig. 6 represents the ideal 2-pole dynamo in which the flux distribution is horizontal and uniform between the poles, when no current flows in the armature.

The *radial* flux density (*i.e.* the density of the flux entering the armature core) is obtained as follows:—

Consider a small area (say 1 square inch) normal to the direction of the flux at AB. The flux crossing this area will be the flux which enters the armature core at BC, which is the area on the surface of the core obtained by projecting

AB horizontally on to it. If OD is the radius drawn to the centre of this area,

$$\text{the area BC} = \frac{1}{\cos \text{ABC}} = \frac{1}{\sin \text{AOF}} \text{ square inches}$$

The flux entering the core through the area BC will thus be β sin AOF lines, if β is the uniform horizontal flux in lines per square inch.

Thus, for points on the circumference of the armature, the radial magnetic density may be represented by a sine curve, exactly as in Fig. 4, but plotted to a horizontal scale, representing the circumference of the armature. Or the field variation may be shown by a rotating vector OA (Fig. 6), whose projections upon a horizontal axis show the successive values of the radial flux density.

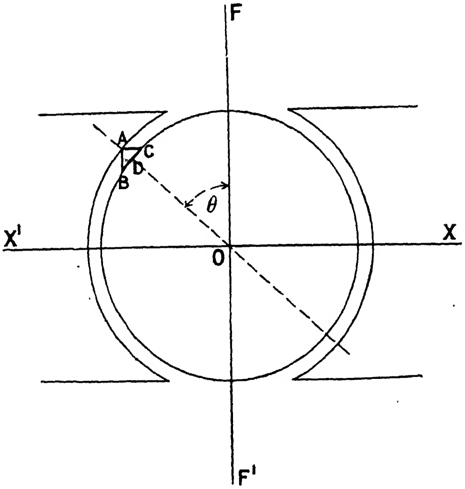

FIG. 6.—Flux in air-gap of simple dynamo.

In a similar manner the field due to the current flowing in the armature conductors may be roughly plotted as a sine wave, whose maximum value occurs along the brush axis, *i.e.* along an axis approximately at right angles to the field axis.

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This curve would be displaced relatively to the previous one by one-quarter of the armature circumference. The rotating line which would represent by its horizontal projections the flux due to armature current, would similarly make an angle of approximately 90° with the line, showing in the same manner the radial flux due to the main field.

In such cases as those just suggested, the base upon which the wave is plotted is one of *space* and not of *time*. In alternating-current work we have often to consider both variations with time and in space.

Representation of Alternating Quantities by Rotating Vectors.—An alternating quantity is represented by a rotating vector in the following manner :—

The rotating vector is a straight line drawn to a predetermined scale so as to represent by its length the maximum value of the alternating quantity. The vector is then conceived to rotate about one of its ends, the other end describing a circle about the fixed end. The moving end is imagined as making $\sim$ revolutions per second, where $\sim$ is the periodicity per second of the alternating quantity represented. The rotation may be taken to occur in either direction, but must always be uniform and in the same direction. *In the following pages the vector will be assumed to rotate in a clockwise direction.*

Since the vector cannot be drawn as actually in rotation, it is drawn in one of the positions through which it is to be conceived as passing, and an arrow-head is added at one end to indicate that this is the rotating end of the line.

The projections of the rotating line on some fixed axis (*e.g.* a vertical axis) are then taken to represent the successive values of the alternating quantity, when referred to the scale originally chosen in fixing the length of the rotating line.

Instantaneous Values.—The actual value of the alternating quantity (as determined from the projection of the rotating line) which corresponds to any position of the rotating vector is generally termed the "instantaneous value of the quantity." With regard to the sense of the instantaneous value, the usual trigonometrical convention is followed as far as possible. Thus, for projections on a horizontal axis, where the arrow-head of the rotating line gives a direction from left to right, the sense is positive, and *vice versâ*. For projections on a vertical axis, instantaneous values are positive when the inclination of the rotating line is such as to give an upward direction to the projection.

In the case of projection on an inclined axis, it is usual to adopt a rule for the sense of the projections, which shall agree as nearly as possible with the above convention.

The instantaneous value of the quantity, as derived from the rotating vector, evidently depends upon the angle at which the rotating vector is drawn with respect to the axis of reference. Thus the same alternating quantity may be represented by the rotating vector drawn at any angle; but each different direction of the vector corresponds to a different instantaneous value of the quantity, *i.e.* it represents the instantaneous value of the same quantity at a different moment of time. As previously stated, we may conveniently take the projections on a *vertical* axis as representing instantaneous values.

The angle between the position of the rotating vector as drawn, and its position when the instantaneous value is zero, determines the *phase* of the alternating quantity. Thus in Fig. 7 the angle POX, giving the angle which OP has moved through from its horizontal position, gives the phase of the quantity. ON represents the instantaneous value of the quantity for the instant for which the vector is drawn. OP represents the maximum value to the same scale.

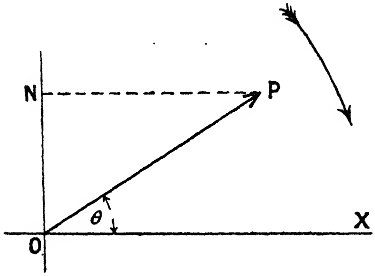

FIG. 7.—A vector.

Treatment of Rotating Vectors.—From what has been said regarding rotating vectors, it appears that a vector may be drawn in any direction, and still represent the same quantity, although the instantaneous value of the quantity will be different for each different inclination of the vector. We shall find that in most actual problems the *relative* inclination of vectors to other vectors on the same diagram is of the utmost importance, while the *actual* inclination is generally chosen from considerations of convenience.

In alternating-current problems, vectors are almost always used in the form of vector diagrams, to show the relations existing between a number of alternating quantities. For instance, we may desire to show the R.M.S. value and relative phase of the current resulting from the application of a voltage having a given R.M.S. value to a certain circuit.

In a vector diagram we are thus concerned with R.M.S. values (and not usually with instantaneous or maximum values), and are concerned with the *phase differences* between the various quantities represented by the vectors (and usually not with the actual phase at any particular instant of time).

We find, therefore, that the three attributes of the vector quantities shown in a vector diagram, which it is of special importance to represent, are: (1) virtual, or R.M.S., value; (2) relative phase; (3) sense.

Now, these are exactly the properties of the alternating quantity which may be obtained from the length, inclination, and position of the arrow-head of the rotating vectors when these vectors are drawn to represent the quantities at any definite *instant of time.*

For example, in Fig. 8, OC, OE represent the current and voltage in a circuit. The length of the lines gives the number of amperes and volts respectively, the angle ϕ gives the difference in phase between current and voltage; the arrow-heads indicate their relative sense.

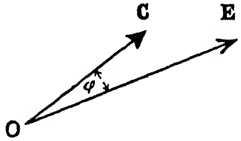

FIG. 8.—Vectors of current and voltage.

When a number of alternating quantities are represented by rotating vectors in the form of a vector diagram, the lengths of the lines usually represent the maximum values of the quantities, and would therefore form the diameters of the polar circles, if the diagram were developed on the lines of Fig. 5.

In the case of simple harmonic quantities, the R.M.S., or virtual, value is a definite fraction of the maximum value, being actually 0·707, or $\frac{1}{\sqrt{2}}$ of the maximum value. The lengths of the lines in the diagram are thus *proportional* to the virtual values of the quantities, and may be taken to represent the virtual values to a scale having a ratio 0·707 : 1 to the scale originally adopted.

In ordinary calculations it is convenient to consider throughout that the lengths of the vectors represent the *virtual* and not the maximum values, and to choose the scale of measurement accordingly.

It has been pointed out earlier (see p. 10) that if we measure instantaneous values as projections on a vertical axis, the angle of inclination of the vector to the horizontal will represent the phase of the quantity (see Fig. 7) at the particular moment for which the diagram is drawn.

Since the diagram represents all the quantities at the same moment of time, the angles between the vectors will show the *relative phase* of the quantities, *i.e.* the difference in phase between them, a complete period being equivalent to an angle of 2π radians, or 360 degrees.

Arrow-heads.—It will be convenient to distinguish, by the character of the arrow-heads, the vectors indicating respectively voltages, currents, and fluxes. We shall adopt the following convention:

Voltages: arrow-head with thin barb;
Currents: „ formed by thick triangular point;
Magnetic fluxes: arrow-head with thin double barb.

The *relative* inclinations of the vectors to one another have definite values determined by the conditions of the problem, so long as the periodicities of the quantities are identical. This is the only case at present under consideration.

The *actual* inclination will depend upon the particular moment chosen for showing the actual phases, and may be varied by changing this. Usually it is convenient so to choose the actual phases of the quantities that one of the vectors of the diagram may be drawn parallel to the axis of reference (*i.e.* usually horizontal) and the actual phase of this quantity is 0°.

The arrow-head at the end of the vector shows in which direction the projection of the rotating line on the axis of reference is to be taken, *i.e.* it indicates whether the projection is positive or negative. The arrow-head thus gives the *sense* of the vector.

It will be found later that vectors representing alternating quantities may be treated exactly like the vectors representing constant quantities as regards rules for addition, subtraction, multiplication, etc.

Vector Diagrams.—It may be well to summarize our discussion of vectors up to this point by the following general statements. In electrical circuits it is often desired to represent the relations between a number of vector quantities which are of different kinds and measured in different units. It is necessary to have a scale for each of the units employed; all quantities measured to the same unit, will then be given by the vector to the same scale. There is, however, no necessary connection between the scales of the different units; indeed, it is often convenient for the same vector to represent more than one quantity to corresponding scales. Thus a vector may represent a current to the scale of amperes and a voltage (current × resistance) to the scale of volts. The scales are in this case chosen to make this possible, and will bear a definite relation to one another determined by the resistance of the circuit.

As previously stated (p. 11) an essential condition of the diagram is that *simultaneous instantaneous values*, or correct *phase relations* of virtual values of alternating quantities must be shown.

From the discussion contained in this chapter it appears that a vector may be used for three purposes :—

(1) To represent the relative magnitude and direction in

space of quantities having a constant value, or an equivalent R.M.S. value;

(2) As a convenient method of obtaining the successive instantaneous values of the alternating quantities;

(3) To represent the relative R.M.S. values and phases of alternating quantities.

Cases often arise which make it convenient to make the same diagram serve more than one of these purposes.

CHAPTER II

VECTOR ALGEBRA

ddition of Vectors.—If we have two lines, such as AB, nd we wish to find the line which is the sum of the ve should place the lines end to end in the same

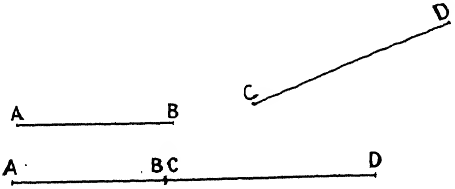

Fig. 9.—Addition of lines.

on, the left-hand end of CD coinciding with the right-end of AB, and thus obtain the resulting line AD. hould then say that the line AD was the "sum" of d CD, meaning thereby that it is the line which has y units of length as AB and CD together.

the case of the addition of vectors we are not concerned he addition of lengths only. Vectors have definite ons, as well as lengths, and cannot be brought into ith one another, when they are to be added together. er to obtain the sum of two vectors we must bring ogether, without altering their inclinations (and having gard to the direction of the arrow-heads), and determine ctor which joins their extremities. This vector is the the two original vectors.

Fig. 10 AB, CD are drawn exactly like the corresponding es in Fig. 9, but with arrow-heads, showing that in

Fig. 10 the lines are vectors. In Fig. 9 the lines represented scalar quantities, *i.e.* quantities having no definite direction —hence the different course which must be followed for the conditions shown in Fig. 10.

In order to add the vector CD to AB, we move CD parallel to itself until C and B coincide, and then join AD. AD is the vector which is the sum of AB, CD. In the triangle ABD, it is to be noted that the arrow-heads of the component vectors point in the same direction round the figure, while the arrow-head of the vector representing their sum is in the opposite direction. This condition is to be observed in all cases of addition.

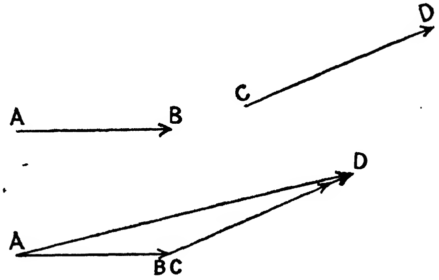

Fig. 10.—Addition of vectors.

We may define the sum of two vectors as being the vector which has the same length and inclination as the line joining the ends of the component vectors, when these are drawn consecutively, so that the beginning of the second coincides with the end of the first.

The sum of two vectors is often called their *resultant.* The term "resultant" indicates the sum of two quantities in which their direction is taken into account; it is thus synonymous with "vector sum."

If A and B represent two vectors, this sum is written

$$A + B$$

It will be noticed that the method of obtaining the sum of two vectors is exactly the same as the method employed in mechanics for obtaining the resultant of two forces; in fact, the "parallelogram law" is the law which we employ for the summation of vectors.

Similarly, the sum of any number of vectors is obtained by drawing the vectors consecutively, and joining the extremities of the system, so as to form a closed figure. The closing line represents the vector which is the sum of the components.

In each case, the arrow-head of the vector giving the sum must point in the opposite direction round the closed figure to the arrow-heads of the component vectors (see Fig. 11).

The *difference* between two vectors A and B can only be looked upon as the sum of the vector A and a vector − B. Thus if in Fig. 12 the vectors A and B have a sum A + B represented by the vector C, their difference A − B will be the vector D, which is the sum of the vector A and the vector B reversed.

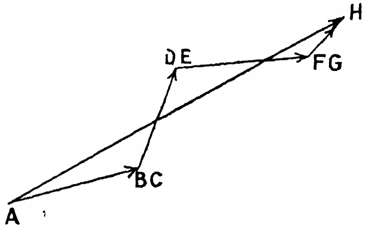

FIG. 11.—Addition of four vectors.

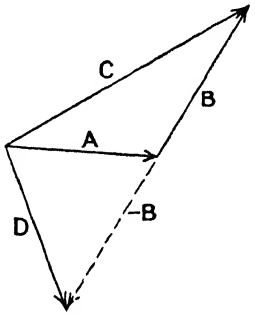

FIG. 12.—Addition and subtraction of vectors.

Addition of Vectors representing Alternating Quantities.—When vectors are employed for showing the magnitude and phase of alternating quantities, the rules just given for obtaining the sum of two or more vectors hold good.

This can be shown by reference to the curves representing the alternating quantities plotted to rectangular co-ordinates.

For example, let us take the case illustrated by Fig. 13, where the two component voltages in an inductive circuit, viz. C*r* and C*x*, are shown, together with the resultant applied voltage.

In the left-hand diagram (Fig. 13) are shown the energy and idle component voltages C*r* and C*x* enclosing a right angle, since they differ in phase by 90°. In this case the vectors are taken as showing *maximum* values.

By rotating the vector C*r* through a series of angles, and for each value of its inclination *θ* projecting its height CN horizontally across to the abscissa on the right corresponding to this angle *θ*, we obtain the sine curve marked C*r*. Carrying out the same process for the vector C*x*, but marking the values of the height of C*x* on the abscissæ corresponding to the values of *θ*, we obtain the second curve C*x* displaced 90° to the left (*i.e.* in advance) of the curve C*r*, *i.e.* displaced by the same angle as that between the vectors.

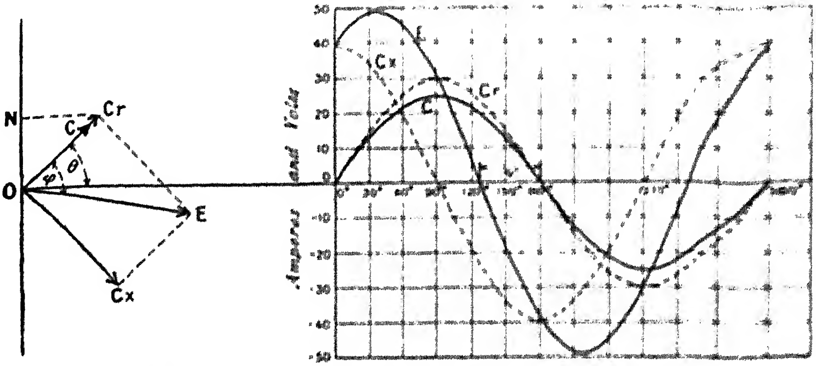

FIG. 13.—Graphic addition of alternating vectors.

If we now add together the ordinates of the voltage curves C*r* and C*x* for each angle (*i.e.* for each moment of time), we obtain the curve E, which shows the successive values of C*r* and C*x* together; in fact, E is the "sum" of the curves C*r* and C*x*.

Now let us find the vector which is the sum of the vectors C*r*, C*x*, by the rules given in the last section, thus obtaining vector E, the diagonal of the parallelogram having C*r* and C*x* as sides. By comparing the vector E with the

ve E we find, by actual measurement, that the vector
esponds, both in regard to successive values of its vertical
ection, and also in regard to its relation in phase, to the
re E. That this must be the case will appear from the
wing reasoning.

The ordinates of the curves Cr and Cx in Fig. 13 represent
essive values of the vertical projections of the rotating
s OCr, OCx, the abscissæ of the curves giving the degrees
gle which the rotating line OCr makes with the horizontal
, *i.e.* the values of θ. Consequently, the curve E will be
um of the vertical projections of Cr and Cx. It is, how-
, evident from the vector diagram that the projections of
line OE will always be equal to the sum of the projections
e lines OCr and CrE whose extremities are joined by OE.
e CrE is equal and parallel to OCx, the projections of OE
qual to the projections of OCr and OCx added together,
hence the curve E will give the values of the projections
E on the same axis and for the same time as the curves
nd Cx give the projections of the vectors Cr and Cx.

Ve may express the fact that the values obtained from
ating vector C are the same as the sum of the values
ined from vectors A and B by writing

$$A + B = C$$

Ve may consequently apply the same rule for addition
ctors representing both fixed and alternating quantities,
the general rule given on p. 17 for the addition of a
er of vectors may be employed for both cases.

he connection between the resultant vector C and its com-
its A, B might have been expressed trigonometrically.
ccessive values of A are given by the expression $a \sin \theta$.
ccessive values of B are given by the expression

$$b \sin (\theta + \phi)$$

ccessive values of C are given by the expression

$$a \sin \theta + b (\sin \theta + \phi)$$

a being the max. value of vector A
b „ „ „ B
ϕ „ phase difference between A and B

Addition of Vectors Commutative.—By applying the rule for the addition of a number of vectors to a given set of vectors, and then repeating the addition, but taking the vectors in a different order, we always arrive at the same resultant. The sum of the vectors is thus unaffected by the order in which the vectors are taken. This is expressed by saying that the addition of vectors is *commutative.*

Symbolic Representation of a Vector.—The graphic representation of alternating quantities by vectors is of the greatest practical use, and many problems may be solved directly with sufficient accuracy by actual measurement of the lines in a diagram. In other cases, especially where quantities of very unequal magnitude occur on the same diagram, or where a resultant is obtained involving the intersection of two lines enclosing a very small angle, exact measurement becomes practically impossible, and it is preferable to use numerical values for the vectors, so as to obtain a result in numerical form.

For this purpose a system of notation must be used giving numerical values for the length and inclination of the lines of a diagram. It is important to remember that this notation does not take the place of the vector diagram, but only expresses the vectors in symbols and enables more accurate values to be obtained, or makes it possible to obtain the same values more rapidly. In many cases it will consequently be desirable to draw a diagram, even where the calculation is carried out algebraically. The diagram may in such cases be only roughly drawn, but will materially assist in the calculation by making the general relations between the factors of the problem clear.

A vector may be completely expressed in terms of its components or projections measured along, and perpendicular to, any fixed axis. It is then denoted simply as the sum of these components. In writing the vector in this form, which is found to be most convenient in general use, it is necessary to indicate by some means that one component is to be measured along the axis of reference, and that the other component is perpendicular to this axis. This is done by Steinmetz by prefixing the letter j to the component vector which is perpendicular to the axis.

Thus the vector of length a, shown in Fig. 14, may be expressed as the sum of its component a_1 along the axis of reference XX′, and of its component a_2 perpendicular to this axis, where

$$a_1 = a \cos \theta$$
$$a_2 = a \sin \theta$$

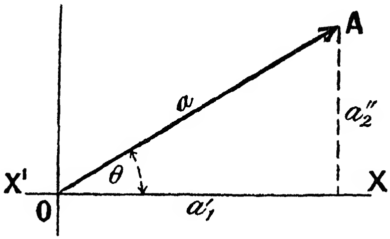

Fig. 14.—Vector and its parallel and normal components.

According to Steinmetz's notation, we should express the vector in the following form:—

$$a_1 + ja_2$$

or $$a \cos \theta + ja \sin \theta = a(\cos \theta + j \sin \theta)$$

This expression indicates that the vector has a component of length $a \cos \theta$ measured along the axis of reference, and a component of length $a \sin \theta$ measured at right angles to it.

For reasons which will be explained later, we propose to use a rather different form of notation in place of the one just described, when writing the vector as the sum of two mutually perpendicular components.

When a vector is written as

V = component along axis + component perpendicular to axis

it is necessary to distinguish between the components. Instead of employing the letter j to denote the perpendicular component, we propose to affix a distinguishing mark to each component, and to employ for this purpose the single and double dashes (′ and ″), as used by engineers for denoting feet and inches.[1]

Components of a vector along the axis of reference will be marked with a single dash (′). Components perpendicular to the axis will have a double dash (″).

We should thus write the vector shown in Fig. 14

$$a_1' + a_2''$$

or

$$(a \cos \theta)' + (a \sin \theta)''$$

This expression indicates that $a_1 = a \cos \theta$ is the component or projection along the axis of reference, and that the component $a_2 = a \sin \theta$ is the component perpendicular to the axis. The + sign indicates that the perpendicular component is to be considered as rotated in a positive or *counter-clockwise* direction from the axis of reference.

It may be noted that the + sign has here exactly the same meaning as on p. 16, where it indicates the addition of two vectors.

The notation just given for a vector may be put in the form of an equation as follows:—

$$A = a_1' + a_2'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$A = (a \cos \theta)' + (a \sin \theta)'' \quad . \quad . \quad . \quad (2)$$

It will be convenient to employ frequently the system of lettering here given, viz.:

---

[1] An advantage of the system of notation here adopted is that j is employed only as an operator, and accordingly always *precedes* the expression whose direction it changes. The dashes indicate units, and are consequently placed *after* the symbol denoting the magnitude of a vector.

A, B, C, etc., capital letters, to indicate a vector or vector quantity.

a, b, c, etc., small letters without dashes, to represent the numerical value or magnitude of the vector.

a', b', c', etc., small letters with single dash, to distinguish components of a vector along the axis of reference.

a'', b'', c'', etc., small letters with double dash, to distinguish components perpendicular to this axis.

With regard to the signs prefixed to the components of a vector, we shall adopt the following rules.

Components along the axis of reference (indicated by a single dash) are positive when directed from left to right, and negative when directed in the reverse sense.

Components perpendicular to this axis (indicated by a double dash) are positive when their sense is such that they are rotated through a right angle in a direction which is counter-clockwise with respect to the reference axis, and negative when rotated in the opposite direction.

It will be convenient to use the terms *parallel* and *normal* components to indicate the components of a vector which are respectively parallel and normal to the axis of reference.

Unit Vectors.—A somewhat more precise definition of the meaning of the suffixes (′) and (″) may now be given.

When engineers employ similar suffixes to denote "feet" and "inches," the single and double dashes indicate units of length. For example, the numeral 23 has no *physical* meaning when written alone; while 23″ indicates a *length of* 23 *units* (inches). The addition of the suffix (″) thus converts a pure number into a physical scalar quantity, viz. into a length.

When employed, as in our case, for indicating the parallel and normal components of a vector, the suffixes again have the significance of units; but they now indicate *units of length along a certain axis.* A number is thus converted into a vector by attaching the suffix to it. This is simply expressed by saying that the symbol (′) represents the unit vector parallel to the axis of reference, and the symbol (″) is the unit vector normal to this axis. It follows that these suffixes are to be considered as definite units, and when they are attached to any number, the resulting expression represents a certain number of units of magnitude of a vector measured along an axis either parallel or normal to the axis of reference.

Derivation of Numerical Value and Inclination from Symbolic Expression. — From Fig. 15, showing the vector $A = a_1' + a_2''$ resolved into its components along and normal to the axis of reference XX′, it is evident that the actual length of the vector is

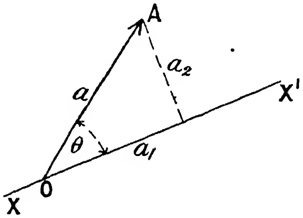

FIG. 15.—Vector showing components parallel and normal to axis of reference.

$$a = \sqrt{a_1^2 + a_2^2}$$

Thus we may always obtain the actual magnitude of a vector by extracting the square root of the sum of the squares of the two components, as given in the symbolic form.

The inclination, θ, of the vector to the axis is given by the relation

$$\tan \theta = \frac{a_2}{a_1}$$

or

$$\theta = \tan^{-1} \frac{a_2}{a_1}$$

In general, the angle of inclination of the vector to the axis of reference is the angle whose tangent is the quotient of the normal component by the component parallel to the axis.

When expressing symbolically a number of vectors in a vector diagram, it is usually convenient to refer them all to the same axis. Generally it is advisable to take this axis parallel to one of the vectors of the diagram, as it simplifies the expression for the vectors parallel to this axis.

Symbolic Representation of Addition.—The graphic addition of vectors has already been discussed (pp. 15 *et seq.*). We have now only to consider the notation for this process when the vectors are given in symbolic form.

In Fig. 16 let A, B, C be vectors of which the resultant is D.

Evidently from the figure the component of D parallel to XX′ is equal to the sum of the components of A, B, and C along the same axis.

Similarly, the component of D normal to the axis is the sum of the normal components of A, B, and C.

Adopting our usual notation for the vectors in terms of their components, the statements just made are equivalent to the equations

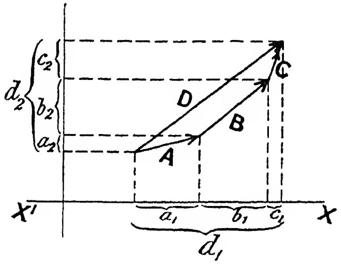

FIG. 16.—Addition of vectors.

$$d_1' = a_1' + b_1' + c_1' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$d_2'' = a_2'' + b_2'' + c_2'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The process of addition may therefore be written in the following form :

$$\begin{aligned} D = A + B + C &= d_1' + d_2'' \\ &= a_1' + b_1' + c_1' + a_2'' + b_2'' + c_2'' \\ &= (a_1 + b_1 + c_1)' + (a_2 + b_2 + c_2)'' \end{aligned}$$

Evidently this may be extended to the addition of any number of vectors.

In carrying out the addition, due regard must be paid to the sign of the component vectors according to the convention given previously (p. 23). The addition of the various components of the same denomination (*i.e.* having the same distinguishing mark ′ or ″) is carried out strictly in accordance with the ordinary rules of arithmetic. In no case can a number of one denomination be added numerically to one of the other denomination.

The general rule for the addition of a number of vectors of form $a_1' + a_2''$ may be written

$$R = \Sigma(a_1') + \Sigma(a_2'')$$

where R is the resultant vector.

Subtraction of Vectors.—As previously explained (p. 17), the process of subtraction is identical with that of addition of vectors having unlike sense. It is consequently not necessary to consider subtraction as distinct from addition.

Resolution of Vectors.—The converse operation to the addition of two vectors to form a single resultant vector, is the resolution of a vector into two component vectors. A special case of such resolution occurs every time we write down the expression for a vector in terms of its parallel and normal components.

A vector may be resolved into two or more component vectors at any inclination to one another, or of any relative magnitude. The only condition to be fulfilled by these components is that the arithmetical sums of their vertical and horizontal components must be respectively equal to those of the original vector.

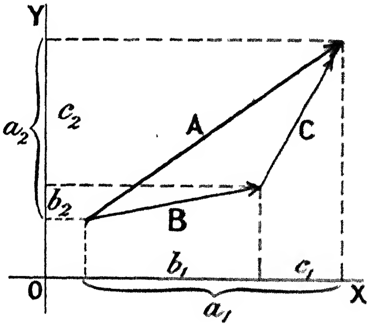

FIG. 17.—Resolution of a vector.

Thus, if

$$A = a_1' + a_2''$$

be the original vector, it may be resolved into any components

$$B = b_1' + b_2'' \quad \text{and} \quad C = c_1' + c_2'', \text{ etc.}$$

in which each of these vectors may have any magnitude and inclination, so long as the conditions are fulfilled that

$$b_1' + c_1' +, \text{etc.} = a_1'$$

and

$$b_2'' + c_2'' +, \text{etc.} = a_2''$$

Resolution of one Vector into Components parallel and normal to another Vector.—As a special instance of the resolution of a vector, we may take the case where it is desired to separate the components of a vector which are respectively parallel and normal to a second vector.

For instance, let it be required to find the components of the vector

$$A = a_1' + a_2''$$

which are parallel and normal to a second vector

$$B = b_1' + b_2''$$

Let the component of A parallel to B be represented by

$$H = h_1' + h_2''$$

and the component of A normal to B by

$$K = k_1' + k_2''$$

There are two cases to be considered: (*a*) when the inclination of A to the axis of reference is less than that of B; (*b*) when the inclination of A is greater.

In case (*a*), illustrated in Fig. 18, by referring to the figure we see the following relations between the component vectors to be found and the vector A :—

$$h_1 + k_1 = a_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$h_2 - k_2 = a_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{h_2}{h_1} = \frac{k_1}{k_2} = \frac{b_2}{b_1} = \text{T} \quad . \quad . \quad (3) \text{ and } (4)$$

where T is the tangent of the angle of inclination of B to the axis of reference = tan β.

Solving these four equations for the four unknowns, h_1, h_2, k_1, k_2, we obtain their values as follows:—

$$h_1 = \frac{a_1 + \mathrm{T}a_2}{1 + \mathrm{T}^2} \quad . \; . \; . \; . \; . \; . \quad (5)$$

$$h_2 = \frac{\mathrm{T}(a_1 + \mathrm{T}a_2)}{1 + \mathrm{T}^2} \quad . \; . \; . \; . \; . \; . \quad (6)$$

$$k_1 = \frac{\mathrm{T}(\mathrm{T}a_1 - a_2)}{1 + \mathrm{T}^2} \quad . \; . \; . \; . \; . \; . \quad (7)$$

$$k_2 = \frac{\mathrm{T}a_1 - a_2}{1 + \mathrm{T}^2} \quad . \; . \; . \; . \; . \; . \quad (8)$$

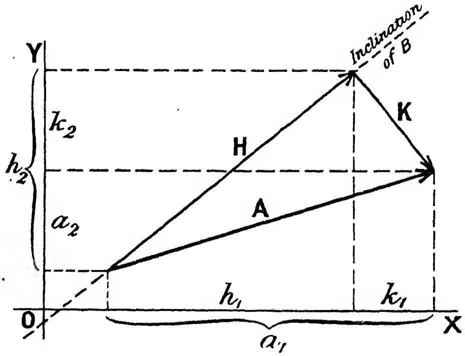

FIG. 18.—Resolution of a vector into components parallel and normal to another vector.

The equations to the parallel and normal components thus become with OX as axis of reference

$$\mathrm{H} = \left(\frac{a_1 + \mathrm{T}a_2}{1 + \mathrm{T}^2}\right)' + \left(\frac{\mathrm{T}(a_1 + \mathrm{T}a_2)}{1 + \mathrm{T}^2}\right)'' \text{ parallel to B, and}$$

$$\mathrm{K} = \left(\frac{\mathrm{T}(\mathrm{T}a_1 - a_2)}{1 + \mathrm{T}^2}\right)' + \left(\frac{\mathrm{T}a_1 - a_2}{1 + \mathrm{T}^2}\right)'' \text{ normal to B}$$

In a similar manner in case (*b*) when A has an inclination greater than B, we have the relations (see Fig. 19)

$$h_1 - k_1 = a_1 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$h_2 + k_2 = a_2 \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

$$\frac{h_2}{h_1} = \frac{k_1}{k_2} = \frac{b_2}{b_1} = \mathrm{T} \quad . \; . \quad (3) \text{ and } (4)$$

By solving as before we have

$$h_1 = \frac{a_1 + Ta_2}{1 + T^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

$$h_2 = \frac{T(a_1 + Ta_2)}{1 + T^2} \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

$$k_1 = \frac{T(a_2 - Ta_1)}{1 + T^2} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$k_2 = \frac{a_2 - Ta_1}{1 + T^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

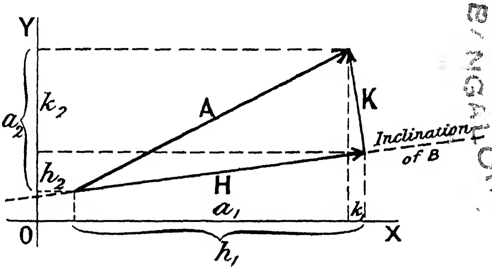

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FIG. 19.—Resolution of a vector into components parallel and normal to another vector.

In this case the component vectors of A are with OX as axis of reference

$$H = \left(\frac{a_1 + Ta_2}{1 + T^2}\right)' + \left(\frac{T(a_1 + Ta_2)}{1 + T^2}\right)'' \text{ parallel to B, and}$$

$$K = \left(\frac{T(a_2 - Ta_1)}{1 + T^2}\right)' + \left(\frac{a_2 - Ta_1}{1 + T^2}\right)'' \text{ normal to B}$$

The values for these components might also have been obtained from the diagram by trigonometry.

The numerical values of these parallel and normal components are more often required. They may be obtained directly from the equations.

Thus in case (a), the numerical value of the vector H

$$= h = \sqrt{h_1{}^2 + h_2{}^2} = \frac{1}{1 + T^2}\left[(1 + T^2)(a_1 + Ta_2)^2\right]^{\frac{1}{2}}$$

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Substituting

$$a_1 = a \cos \alpha$$
$$a_2 = a \sin \alpha$$

where α is the inclination of A to the axis of reference, β, as before, being the inclination of B to the axis, we obtain

$$h = \sqrt{\frac{a^2(\cos \alpha + \tan \beta \sin \alpha)^2}{1 + \tan^2 \beta}} = a(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$
$$= a \cos (\beta - \alpha)$$

Similarly, the value of the component normal to B is

$$k = \frac{1}{1 + T^2}[(1 + T^2)(\tan \beta \cos \alpha - \sin \alpha)a^2]^{\frac{1}{2}}$$
$$= a(\sin \beta \cos \alpha - \sin \alpha \cos \beta) = a \sin (\beta - \alpha)$$

In case (*b*) we should similarly find the numerical values to be

$$h = a \cos (\alpha - \beta)$$
$$k = a \sin (\alpha - \beta)$$

The same results may be obtained more directly without finding the general equation to the component vectors.

Let the vectors be as before

$$\mathrm{A} = a_1' + a_2''$$
$$\mathrm{B} = b_1' + b_2''$$

Then referring to the figure, it is seen that the component of A in phase with B is

$$\sqrt{a_1^2 + a_2^2} \cos (\alpha - \beta) = a \cos (\alpha - \beta)$$

and the component of B in phase with A is

$$\sqrt{b_1^2 + b_2^2} \cos (\beta - \alpha) = b \cos (\beta - \alpha)$$

where α is the angle made by A with the axis of reference, and β is the corresponding angle of B, so that

$$\alpha = \tan^{-1}\frac{a_2}{a_1} \qquad \text{and } a = \text{numerical value of A}$$

$$\beta = \tan^{-1}\frac{b_2}{b_1} \qquad b = \quad ,, \quad ,, \quad \text{B}$$

Similarly, the component of A normal to B is

$$\sqrt{a_1^2 + a_2^2}\sin(\alpha - \beta) = a\sin(\alpha - \beta)$$

and the component of B normal to A is

$$\sqrt{b_1^2 + b_2^2}\sin(\beta - \alpha) = b\sin(\beta - \alpha)$$

It is to be remembered that, according to our convention, angles measured in a counter-clockwise direction are positive, and an angle which has a minus sign before it is measured in a clockwise direction.

Exponential Form of Notation.—Vectors may also be expressed in another form, which it is convenient to use in certain cases. The representation by exponential functions is derived from the expressions for sine and cosine of an angle as expanded series.

The connection between the usual trigonometrical form and the exponential form of notation for a vector may be shown as follows.

If in the usual series

$$\varepsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \dots$$

we give to x the value $j\theta$, the series becomes

$$\varepsilon^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{\underline{|2}} + \frac{(j\theta)^3}{\underline{|3}} + \frac{(j\theta)^4}{\underline{|4}} + \dots$$

Substituting the value $j^2 = -1$, we obtain [1]

---

[1] It is shown on p. 44 that the value of j^2 is -1.

$$\varepsilon^{j\theta} = 1 + j\theta - \frac{\theta^2}{\underline{|2}} - \frac{j\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + \frac{j\theta^5}{\underline{|5}} + \dots$$
$$= 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \dots + j\left(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \dots\right)$$

Now the usual expansions of $\sin\theta$ and $\cos\theta$ give the following series:—

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \frac{\theta^9}{\underline{|9}} - \dots$$
$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \dots$$

Hence it is apparent that

$$\varepsilon^{j\theta} = \cos\theta + j\sin\theta$$

and the expression for the vector as the sum of its co-ordinates may be otherwise written

$$A = a(\cos\theta' + \sin\theta'') = a(\cos\theta + j\sin\theta) = a\varepsilon^{j\theta}$$

where j, besides its algebraic value of $\sqrt{-1}$, has the signification that the term containing it is measured normal to the axis of reference.

In the exponential form of a vector, $\underline{a\varepsilon^{j\theta}}$, it is evident that a represents the length of the vector, and θ its inclination to the axis of reference.

Example I.—A certain 300 K.W. two-phase alternator has an armature 14 feet in diameter, and 64 poles whose faces have been designed to give as nearly as possible a sinusoidal flux distribution. The armature has 256 slots, and each slot contains 8 conductors; what is the voltage per phase, and what voltage would the machine give as a single-phase generator if all the coils of both phases were connected in series?

The maximum air-gap density is 60,000 lines per square inch, the speed of the machine 93·7 revolutions per minute, and the axial length of the armature face 6″.

The width of a slot will, of course, be small compared to the width of the pole-arc; so that we may consider all the conductors in one slot as generating at any given instant equal E.M.Fs.

Since there are 256 slots total, the slots per phase = 128; *i.e.* corresponding to each pole there are 4 slots, two in one phase and two in the other. These slots are therefore $\frac{\pi}{4}$ radians or 45 electrical degrees apart.

The arrangement is shown in the diagram, Fig. 20.

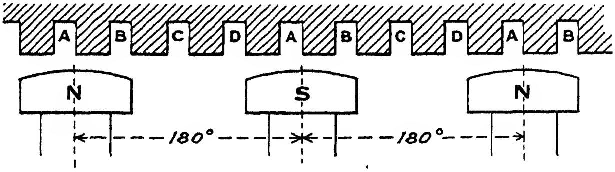

FIG. 20.—Relation between slots and poles of alternator.

Let the maximum E.M.F. generated by a conductor in any such slot as A be called E_1.

Then E_1 = length of armature face in inches × circumferential velocity of poles in inches per second × maximum flux density per square inch ÷ 10^8

$$= 6'' \times \pi \times 14 \times 12 \times \frac{93\cdot7}{60} \times \frac{60{,}000}{10^8}$$

$$= 3 \text{ volts}$$

There are 64 slots similarly placed, and 8 conductors per slot, so that the maximum value of the E.M.F. due to these conductors = 64 × 3 × 8 = 1536 volts.

Now since the flux is sinusoidal, the voltage is sinusoidal and may be represented by a rotating vector A.

The other 64 slots in the same phase generate a like voltage, but being spaced 45° from slots A, their E.M F. vector is similarly spaced in time. This voltage is therefore represented by a second vector B, where angle AOB = 45°.

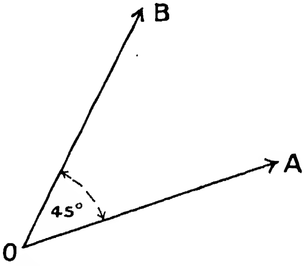

FIG. 21.—Diagram of voltages for one phase.

The sum of these two is the maximum voltage per phase, and this may be obtained either by drawing a diagram to scale and measuring the resultant, or in symbolic notation as follows:—

Choose vector OA as the axis of reference.

Then we may write $A = a'$

$$B = b_1' + b_2''$$

$$\tan\theta = \frac{b_2}{b_1} = \tan 45^\circ = 1$$

whence $b_1 = b_2$

But $\sqrt{(b_1^2 + b_2^2)} = a = 1536$

that is, $\sqrt{2b_1^2} = 1536$

$$b_1 = b_2 = \frac{1536}{\sqrt{2}} = 1089$$

and $A + B = a' + b_1' + b_2''$

$$a + b = \sqrt{(a + b_1)^2 + b_2^2} = 2800 \text{ volts}$$

The new vector A + B is at an angle to A whose tangent

$$= \frac{b_2}{a + b_1} = \frac{1089}{2625} = 0{\cdot}414$$

i.e. an angle of $22\frac{1}{2}^\circ$.

The maximum voltage per phase is then 2800 volts, and its R.M.S. value $\frac{2800}{1{\cdot}41} = 1980$ volts.

The two phases of a two-phase circuit are equal and mutually perpendicular. Calling them C and D, we have

$$C = D = 1980 \text{ R.M.S. volts}$$

If all the coils of this machine were connected in series, we should find the resulting voltage as follows :—

Let D be along the axis of reference.

Then $D + C = d' + c''$

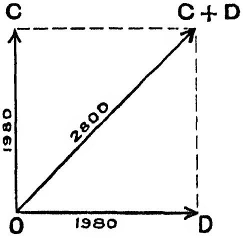

FIG. 22.—Diagram of voltages, with two phases in series.

Numerical value of $d' + c'' = \sqrt{c^2 + d^2} = \sqrt{(1980)^2 + (1980)^2}$

$$= \sqrt{2\,.\,(1980)^2} = \sqrt{2} \times 1980 \text{ volts}$$

$$= 2800 \text{ volts}$$

Also C + D is at an angle θ to D such that

$$\tan\theta = \frac{c}{d} = 1$$
$$\theta = 45^\circ$$

Corollary I.—As a corollary to this problem we see that for a given generator with a given number of conductors on the armature, the maximum voltage is produced by a concentration of the conductors per pole. For, as shown above, when there are eight conductors per slot and one slot per pole, the R.M.S. voltage is $\frac{1536}{\sqrt{2}} = 1089$; with eight conductors per slot and two slots per pole 45° apart, the R.M.S. voltage is 1980; with the two phases in series, *i.e.* four slots per pole 45° apart, the voltage is 2800 volts; *i.e.* for the last case, though we have four times the number of conductors per pole compared with the first case, yet the voltage is only increased in the ratio of $\frac{2{\cdot}8}{1{\cdot}089}$ or 2·57 : 1. If, however, all these conductors had been concentrated into one narrow slot we should obtain evidently 4 × 1089 or 4356 volts. This arrangement has disadvantages which we need not enter upon here, but the case is interesting as emphasizing the difference between vector addition and arithmetical addition. The difference is illustrated in Fig. 23, where OA, AB, etc., each represent an E.M.F. vector of value 1089 volts. When the conductors are distributed in four slots we get the resultant OD in Fig. 23*a* = 2800 volts; but when they are all in one slot we get OD (Fig. 23*b*) = 4 × 1089 = 4356 volts.

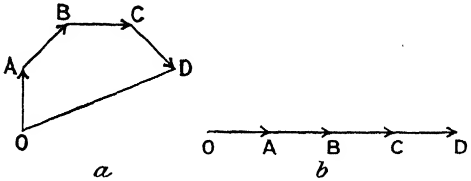

FIG. 23.—Armature voltage. *a*, four slots per pole; *b*, one slot per pole.

Corollary II.—If the conductors had been uniformly distributed around a smooth core instead of placed in slots, then the resultant voltage would have been nearly $\frac{2}{\pi}$ times the number of conductors × voltage per conductor.

For in this case, instead of four vectors OA, AB, etc., differing in phase by 45°, we get 32 smaller vectors (each = 136 volts) differing in phase by $\frac{180}{\text{conductors per pole}}$ degrees, *i.e.* by 5·63°. This is shown in Fig. 24, and it will be seen that the vector polygon is practically a semicircle with the resultant OD as diameter. With an infinite number of conductors it is actually a semicircle, and in practice, where such uniform distribution is adopted, as in a rotary converter, it is usually near enough to assume this form.

From the figure for this case

$$\text{length of arc of semicircle} = 32 \times 136 = 4356 \text{ volts nearly}$$

$$\text{resultant} = \text{diameter} = \frac{2}{\pi} \times 4356 = 2775 \text{ volts nearly}$$

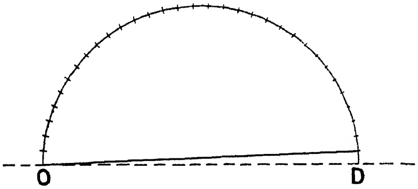

FIG. 24.—Armature voltage with distributed winding.

Hence the limiting voltage conditions for this machine as a single-phase alternator with the constants given are as follows:—

All conductors bunched in narrow slots, 4356 volts.
,, ,, uniformly distributed over armature face, 2775 volts.

As an example of the use of a vector diagram to show the *space relations* between vector quantities, we may consider the following problem.

EXAMPLE II.—Relation between ampere-turns in stator and rotor of an induction motor.

The stator of a three-phase 6-pole motor has all its coils placed in series, and is then excited by means of a constant continuous current of 4·5 amperes, giving approximately 3000 total stator ampere-turns. The winding is such as to produce a practically sinusoidal flux distribution in the air-gap.

If the short-circuited rotor of this machine is driven round at 40 R.P.M. a voltage will be generated in the rotor conductors. It is calculated that this voltage will produce a current such that the total average rotor ampere-turns will be 600.

Give a vector diagram showing the relationship between rotor and stator ampere-turns under these conditions, and find the approximate rotor C.R. drop per pole in terms of the rotor E.M.F. when the rotor is open circuited.

To make such a case as this more clear we shall first consider a bipolar machine.

We may assume that in this case the flux due to the stator windings may be taken as forming a uniform horizontal field, so that the voltage induced in the rotating conductors of the rotor will be sinusoidal (*cf.* p. 8). A vector drawn through the polar axis may now be taken to represent this flux in magnitude and direction.

As the rotor revolves, an alternating voltage is induced in each of the conductors as it cuts the field in the air-gap. Each conductor of the rotor will consequently carry an alternating current, but the *space distribution* of current in the rotor considered as a whole will be constant, and will be such that it will produce a flux (due to the rotor ampere-turns) exactly at right angles to the flux in the air-gap. This air-gap flux is, however, no longer the flux due to the stator ampere-turns alone, but will be the sum of the stator and rotor fluxes, so soon as the rotor carries a current.

The actual air-gap flux is thus the vector sum of the stator flux, and of a flux (due to the rotor ampere-turns) perpendicular to the resultant air-gap flux.

The relation between these fluxes must therefore be as shown in Fig. 25, where A_1 is the flux due to the stator ampere-turns, C_1 is the air-gap flux, and B_1 perpendicular to this is the flux produced by the rotor current.

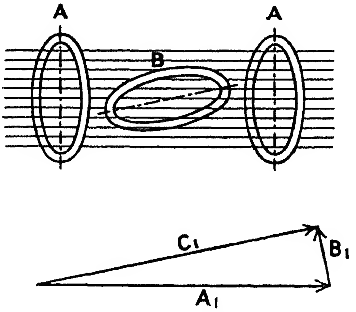

FIG. 25.—Diagram showing relation between ampere-turns in induction motor.

Since these fluxes are all produced in the same magnetic circuit, they will be proportional to the ampere-turns producing them. The vector diagram may thus be taken to show the number and line of action of the ampere-turns producing the three fluxes A, B, D as well as the fluxes themselves.

Let the coils be represented diagrammatically in Fig. 25, where A, A are the stator windings, and B is a winding of the rotor which coincides with the plane of its resultant current.

Vector A_1 in the lower diagram perpendicular to the planes of A, A represents the stator ampere-turns. Vector B_1 perpendicular to the plane of B represents the rotor ampere-turns, while the vector C_1, which is the vector sum of A_1 and B_1, must be parallel to the coil B, since it is by the movement of the coil across the flux caused by C that the E.M.F. is generated, which causes the current to flow through the resistance of the coil.

These vectors represent the corresponding ampere-turns in magnitude and direction, so that we may obtain the value of the resultant ampere-turns producing the air-gap flux from the equation

$$A_1 = \sqrt{B_1^2 + C_1^2}$$

Returning now to the 6-pole motor of the original example, the stator ampere-turns per pole

$$= \frac{3000}{6} = 500 = A_1$$

and the rotor ampere-turns per pole

$$= \frac{600}{6} = 100 = B_1$$

Hence $$\begin{aligned} C_1 &= \sqrt{(500)^2 - (100)^2} \\ &= 490 \end{aligned}$$

The flux is proportional to this vector, so that at this speed the rotor C.R. drop is $\frac{490}{500}$ of that E.M.F. which would be generated if the rotor were stationary and on open circuit.

The above example is of importance as representing practically that which takes place in the case of a 3-phase motor. The speed given above would be the slip below synchronism, and the E.M.F. just calculated would be the back E.M.F. produced in the rotor by the rotating field + the C.R. drop (when the machine is on load) calculated in terms of the open-circuit rotor E.M.F.

Similarly the case may stand for a single-phase induction motor, and in such a case the rotor ampere-turns at right angles to the stator field are those which produce the second component of the resultant rotating field.

Of course, the angle between the axis of coil B and the axis of coils A represents the angle between the stator-coil axis and the rotor-field axis in a bipolar machine. In a 6-pole machine it would actually have $\frac{1}{3}$ of this value.

CHAPTER III

MULTIPLICATION OF VECTORS

Multiplication of a Vector by a Number.—The effect of multiplying a vector by a simple[1] number is to alter its length without affecting its inclination.

It follows that when a vector is multiplied by a simple number, both horizontal and vertical components of the vector must be considered to be multiplied by the number. Thus in Fig. 26, if

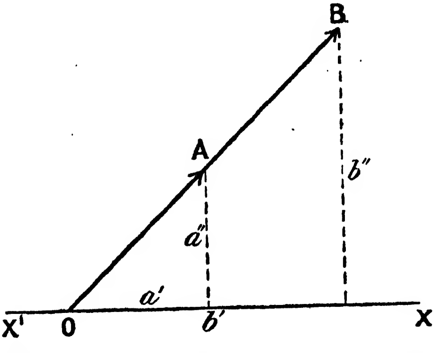

Fig. 26.—Graphic representation of B = 2A.

$$B = 2 \times A$$

it is evident that the components of B must satisfy the conditions

$$b' = 2a'$$

and $$b'' = 2a''$$

---

[1] Numbers which are not "simple" are the "complex" numbers which are considered later. Simple numbers include all real numbers and fractions.

We may accordingly write the multiplication of the vector

$$A = a' + a''$$

thus $$B = 2 \times A = 2a' + 2a'' = 2(a' + a'')$$

Division by a simple number may be treated in exactly the same way, since division by a number is only multiplication by its reciprocal.

Non-inductive Circuit.—We may take as an example the case of a current flowing in a non-inductive circuit. Here we have for the voltage applied to the circuit

$$\underset{\text{(vector)}}{\text{Voltage}} = \underset{\text{(vector)}}{\text{current}} \times \underset{\text{(simple number)}}{\text{resistance}}$$

The resistance determines the ratio of the magnitude of the current vector to that of the voltage vector, but introduces no change in the phase of the vectors, so that the current vector has the same phase as the voltage vector.

If the current vector is written in the form

$$C = c_1' + c_2'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then the voltage is given by

$$E = rC = rc_1' + rc_2'' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where r is the resistance through which C flows.

From the equation (2) we may obtain the numerical value of the voltage by the usual method

$$e = \sqrt{(rc_1)^2 + (rc_2)^2} = r\sqrt{c_1^2 + c_2^2}$$
$$= rc \text{ volts}$$

where c is the numerical value of the current in amperes.

Conversely, we may obtain the current from a known voltage applied to a circuit of resistance r ohms.

$$\underset{\text{(vector)}}{\text{Current}} = \underset{\text{(vector)}}{\text{voltage}} \times \underset{\text{(number)}}{\frac{1}{\text{resistance}}}$$

If the voltage is given as

$$E = e_1' + e_2''$$

then
$$C = \frac{E}{r} = \frac{e_1'}{r} + \frac{e_2''}{r} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By calculating the numerical relations as before, we get the usual expression for Ohm's law

$$c = \frac{e}{r}$$

Since division by a number is equivalent to multiplication by the reciprocal of the number, we may write the above equation

$$C = \frac{E}{r}$$

in the form

$$C = E \times g = ge_1' + ge_2'' \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where g is the reciprocal of r, and is called the *conductance*[1] of the circuit.

It is evident that equations (3) and (4) will express exactly the same relation. It will be found later that the second form is often the more convenient.

Magnetic Flux due to Magnetizing Ampere-turns.—The relation between the flux in a magnetic circuit and the magnetizing current may be put in the form

$$F = ht\sqrt{2}C$$

where h is a constant of the magnetic circuit, representing the flux per ampere-turn (we neglect variation in permeability).

t is the number of turns of the magnetizing winding.

C is the R.M.S. current in the winding.

F is the maximum flux.

---

[1] The circuit is here assumed to have no reactance. The conductance of a reactive circuit is shown later to have the value (see p. 58)

$$g = \frac{r}{r^2 + x^2}$$

If the current be written in the form

$$C = c_1' + c_2''$$

in which c_1 and c_2 are numbers of known value, the expression for the flux will be

$$F = ht\sqrt{2}C = ht\sqrt{2}c_1' + ht\sqrt{2}c_2'' \quad . \quad . \quad . \quad (5)$$

from which the phase and magnitude of the flux due to a given current are determined. This relation is more fully considered in Chapter IV. (see p. 74 *et seq.*).

The converse case, in which the current to produce a given flux is to be found, is of frequent occurrence, and is dealt with in connection with the magnetizing currents of transformers and induction motors in subsequent chapters.

Multiplication by j.—Cases occur in which the direction, as well as the magnitude, of a vector are changed by multiplication. For instance, when a current of c amperes flows in a circuit having negligible resistance, but a reactance of x ohms, we obtain the numerical value of the voltage applied to the circuit as $c \times x$ volts. The voltage vector will differ from the current vector not only in length, but will be rotated relatively to it through 90°, since there will be this difference of phase between the two vector quantities.

In order to indicate a change in the direction of a vector, the symbol j has been adopted, its signification being that *when any vector is multiplied by j, that vector is thereby rotated through 90° in a counter-clockwise direction.*

Thus in the example suggested above, the complete relation between the current and the voltage opposing the applied voltage in a reactive circuit would be expressed

$$-E = jxC \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where E represents the voltage applied.

Multiplication by j affects the inclination but not the length of a vector.

When a vector is multiplied by j, the whole vector is rotated through a right angle.

if A is a vector with components a_1' and a_2''

$$jA = j(a_1' + a_2'') = ja_1' + ja_2''$$
$$= a_1'' - a_2' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

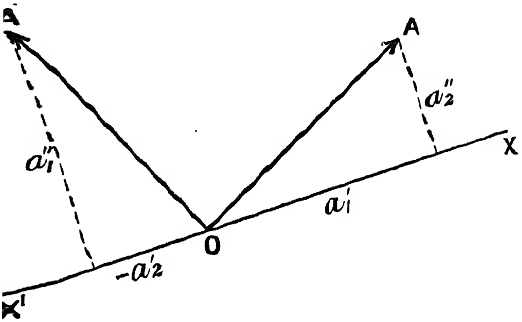

FIG. 27.—Effect of multiplication by *j*.

$_1$ along the axis when rotated through 90° becomes l to the axis. Also, a_2 normal to the axis when hrough 90° becomes parallel to the axis, and if it iously *positive* (*i.e.* if it previously made a right th the axis measured in the counter-clockwise from the axis) it will now be directed in a *negative* ng the axis.

the considerations just stated we may write down ving rules with regard to sign :—

$$ja' = a''$$
$$-ja' = -a''$$
$$ja'' = -a'$$
$$-ja'' = a'$$

e vector A in the above case had been multiplied by ad of by *j*, we should have obtained

$$-jA = -j(a_1' + a_2'') = -ja_1' - ja_2''$$
$$= -a_1'' + a_2' = a_2' - a_1'' \quad . \quad . \quad . \quad . \quad (8)$$

vector is multiplied by *j* twice in succession, it will be rotated through 90° twice in the same sense, *i.e.* through 180° in all. It thus appears that multiplication by $j \times j$ (which we may conveniently write j^2) has the effect of reversing the sense of a vector, since a rotation through 180° is equivalent to a change of sign.

Thus multiplication by j^2 has exactly the same effect as multiplication by -1. For this reason it is convenient to treat j^2 and -1 as being equivalent, and to write

$$j^2 = -1$$

If now we take the square root of these equal quantities we obtain

$$j = \sqrt{-1}$$

i.e. j is the imaginary root of -1. This connection between j and $\sqrt{-1}$ it is convenient to employ in certain methods of calculation. The only physical interpretation which we can put upon it is that, when considering lengths measured along any given axis, all lengths measured at right angles to it are to be looked upon as imaginary. Thus the meaning of an imaginary part in the expression for a vector is that this part is to be measured along a different axis from the rational part. As pointed out later, the symbol j often distinguishes the "idle" or "wattless" component of a current or voltage from the energy component. In this case, the term "imaginary" is equivalent to "wattless."

Multiplication by j produces counter-clockwise rotation through 90°.

Multiplication by j^2 or -1 produces counter-clockwise rotation through 180°.

Multiplication by j^3 or $-j$ produces counter-clockwise rotation through 270°, or clockwise rotation through 90°.

If a vector is multiplied by kj where k is a simple number, the result is a vector rotated through a right angle (due to the operation of j) and altered in length through multiplication by k.

Thus

$$\begin{aligned} jkA &= jk(a_1' + a_2'') = jka_1' + jka_2'' \\ &= ka_1'' - ka_2' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9) \end{aligned}$$

Reactance and Susceptance.—Instances of the multiplication of a vector by an expression of the form jk frequently occur in connection with the relations between the current and voltage of inductive circuits.

Some important cases are given below.

It will be seen from them that an inductive circuit has a reactance which is positive ($x = 2\pi \sim l$), while a circuit with capacity has a negative reactance $\left(x = \dfrac{-1}{2\pi \sim K}\right)$. A circuit with both inductance and capacity may have either a positive or negative reactance.

Circuit with Inductance.—In the ideal case of an inductive circuit through which a current flows without energy being expended, the numerical relation between the applied voltage and current is given by the equation

$$e = cx$$

where x = the reactance = $2\pi \sim l$
l = coefficient of self-induction of the circuit in henrys
$\sim$ = frequency of the applied voltage

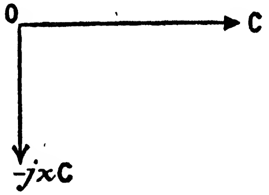

FIG. 28.—Current and voltage in a partly inductive circuit.

The reactance has the effect of making the current lag behind the applied voltage in phase by 90°. The complete relation between current and applied voltage is consequently given symbolically

$$E = -jxC$$

showing that the voltage leads the current in phase, or if the complete expression for C is $c_1' + c_2''$, we have

$$E = -jxC = -jx(c_1' + c_2'') \quad . \quad . \quad . \quad (10)$$

The symbolic expression for the reactance of an inductive circuit is thus $-jx$.

Circuit with Capacity.—When no losses occur in a circuit having capacity, the numerical relation between applied voltage and charging current is given by the equation

$$e = -cx = \frac{c}{2\pi \sim \mathrm{K}}$$

where reactance $= x = \dfrac{-1}{2\pi \sim \mathrm{K}}$

$\mathrm{K} =$ capacity in farads

$\sim =$ frequency of the applied voltage

The charging current leads the voltage in phase by 90°, since the reactance in this case has a negative value. The connection between voltage and current is shown as before by multiplying the reactance by $-j$. Thus

$$\mathrm{E} = -jx\mathrm{C} = -jx(c_1' + c_2'') \quad . \quad . \quad . \quad (11)$$

where $c_1' + c_2''$ is the complete form of the expression for the current, and x has a negative value.

Circuit with Inductance and Capacity.—It has been shown that when a current flows in an inductive circuit the voltage necessary to overcome the reactance is

$$\mathrm{C} \times (-jx_1)$$

where $x_1 =$ reactance due to the inductance. Also in a circuit possessing capacity the voltage required to overcome the reactance x_2 due to capacity is

$$\mathrm{C} \times (-jx_2)$$

Where reactances due to both inductance and capacity are present, there must be voltages overcoming both reactances x_1 and x_2. The total voltage overcoming reactance will therefore be the sum of these, *i.e.*

$$\text{Voltage overcoming reactance} = \mathrm{E} = \mathrm{C}(-jx_1 - jx_2)$$
$$= -j\mathrm{C}(x_1 + x_2) = -j\mathrm{C}x$$

where

$$x = \text{total reactance of circuit}$$
$$= x_1 + x_2$$
$$= 2\pi \sim l - \frac{1}{2\pi \sim \mathrm{K}}$$

which may be either a positive or negative number.

Division by *j*.—If we divide a quantity by j, there results a fraction with a denominator which is "imaginary," since it will be multiplied by the imaginary quantity j, or $\sqrt{-1}$. Such a fraction must be rationalized before any meaning can be assigned to it. This may be illustrated by the following example.

In a circuit having a reactance of x ohms and an applied voltage E, the current is given by

$$\mathrm{C} = \frac{\mathrm{E}}{-jx}$$

In order to rationalize this fraction, multiply numerator and denominator by j;

$$\mathrm{C} = \frac{j\mathrm{E}}{-j^2x} = +\frac{j\mathrm{E}}{x} = \frac{j}{x}(e_1' + e_2'')$$
$$= \frac{1}{x}(e_1'' - e_2') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The reciprocal of the reactance of a circuit is termed its *susceptance*,[1] and is usually denoted by the symbol b.

Hence, writing the reactance x we have

$$\text{susceptance} = \frac{1}{x} = b$$

and the relation between current and voltage in a reactive circuit (without resistance) may be written in either of the forms

---

[1] It is here assumed that the circuit has no resistance. As shown later, the susceptance is $\frac{x}{r^2 + x^2}$ in a circuit having both resistance and reactance.

$$E = -jxC \quad . \quad . \quad \text{(equation (10))}$$

or

$$C = jbE \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Example 1.—A condenser having a capacity of 20 microfarads is connected to a 500-volt circuit having a frequency of 50 cycles per second. Assuming the wave form of the voltage to be sinusoidal, calculate the charging current taken by the condenser.

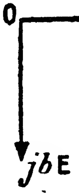

Fig. 29.—Relation between applied voltage and charging current.

In this example the susceptance of the circuit formed by the condenser is

$$\begin{aligned} b &= -2\pi \sim K \\ &= -\frac{2\pi \times 50 \times 20}{10^6} \\ &= \frac{-6{\cdot}283}{10^3} \\ \text{Current} = C &= jbE \end{aligned}$$

or, if the voltage vector is taken as axis of reference

$$\begin{aligned} C &= jbe' \\ &= j\left(\frac{-6{\cdot}283 \times 500}{1000}\right)' = -j\,3{\cdot}1416' = -3{\cdot}1416'' \text{ (see Fig. 29)} \end{aligned}$$

Example 2.—A 150 K.V.A. transformer for 50 ~ having a maximum flux in the core of 2,000,000 lines, has a primary susceptance of 0·000434, and 676 primary turns. Find the value of the magnetizing current taken at no load.

In a transformer the magnetizing current is proportional to the voltage which is produced by the flux set up by the magnetizing current, *i.e.* the magnetizing current is proportional to that part of the total applied voltage which balances the induced back E.M.F. of the transformer. This voltage we shall call E_1 in the present and later problems. Here its value is given by the ordinary E.M.F. formula

$$\begin{aligned} e_1 &= 4{\cdot}44 \times \sim \times \text{maximum flux} \times \text{primary turns} \times 10^{-8} \\ &= 4{\cdot}44 \times 50 \times 2 \times 10^6 \times 676 \times 10^{-8} \\ &= 3000 \text{ volts} \end{aligned}$$

The exact relation between the magnetizing current and the voltage E_1 is

$$C_m = jbE_1$$

If we take the vector E_1 as the axis of reference

$$\begin{aligned} C_m &= jbe_1' \\ &= j(0{\cdot}000434 \times 3000)' \\ &= j1{\cdot}3 = 1{\cdot}3'' \end{aligned}$$

The magnetizing current is, therefore, 1·3 amps., 90° in phase behind the voltage (see Fig. 30).

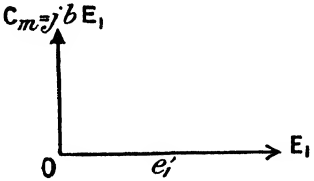

FIG. 30.—Relation between magnetizing current and voltage overcoming induced volts.

Multiplication by Complex Number.—The example just given is a special case of multiplication by a complex number.

A *complex number* has in general both real and imaginary parts, *i.e.* it has a component containing the imaginary quantity j and also a component which is real, *i.e.* which does not contain j.

According to the ordinary rules of algebra, multiplication by a quantity of the form $(a + b)$ gives a product which is the sum of the separate products obtained by multiplication by a and b separately. The same rule holds in the case of the multiplication of vectors. Thus, since a complex number will generally be the sum of a real and an imaginary quantity, the product obtained by multiplication by a complex number may be arrived at by multiplying first by the real part, then by the imaginary part, and subsequently adding the products together.

Let $A = a_1' + a_2''$ be a vector, and $h + jk$ a complex number by which the vector A is to be multiplied.

Multiplication by the real part h gives

$$hA = ha_1' + ha_2'' \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Multiplication by the imaginary part gives as the result the perpendicular vector

$$jkA = jka_1' + jka_2'' \qquad . \quad . \quad . \quad . \quad . \quad (15)$$
$$= ka_1'' - ka_2'$$

Adding together the products (14) and (15) we obtain

$$(h + jk)A = ha_1' - ka_2' + ha_2'' + ka_1'' \qquad . \quad . \quad (16)$$

This is a vector having a parallel component

$$ha_1' - ka_2'$$

and a normal component

$$ha_2'' + ka_1''$$

Its inclination to the axis of reference is the angle

$$\theta = \tan^{-1} \frac{ha_2 + ka_1}{ha_1 - ka_2}$$

and its length is

$$\sqrt{(ha_2 + ka_1)^2 + (ha_1 - ka_2)^2}$$

We see that multiplication by a complex number changes both magnitude and inclination of a vector.

The result of multiplying the vector

$$A = a_1' + a_2''$$

by the complex number $h + jk$ may be shown in another manner.

Multiplying separately the parallel and normal components of A by the number $h + jk$, the parallel component becomes

$$(h + jk)a_1' = ha_1' + ka_1''$$

This is evidently a line making an angle

$$\tan^{-1} \frac{a_1k}{a_1h} = \tan^{-1} \frac{k}{h}$$

with the axis, *i.e.* with its previous direction. Further, the length of this line is now $\sqrt{(a_1h)^2 + (a_1k)^2} = a_1\sqrt{h^2 + k^2}$, *i.e.* its previous length multiplied by $\sqrt{h^2 + k^2}$.

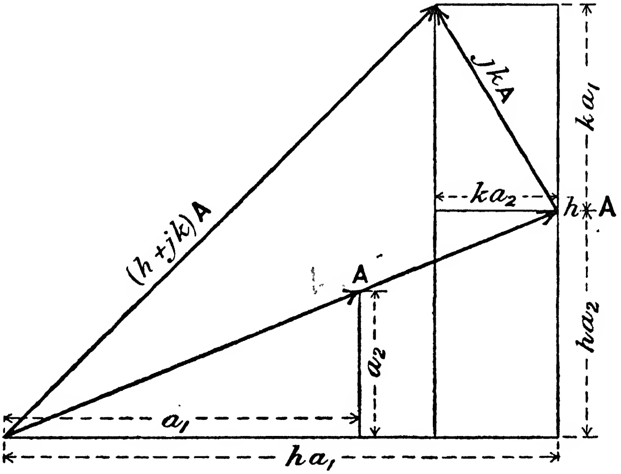

FIG. 31.—Multiplication of a vector by a complex number.

Similarly, the normal component of the vector becomes by the multiplication

$$(h + jk)a_2'' = ha_2'' - ka_2'.$$

This makes an angle $\tan^{-1}\dfrac{-h}{k}$ with the axis, or an angle of $\tan^{-1}\dfrac{k}{h}$ with the normal, *i.e.* its previous position. The length of the vertical component is similarly seen to be equal to its previous length multiplied by the factor

$$\sqrt{h^2 + k^2}$$

Thus both horizontal and vertical components of the vector A are rotated through the same angle $\tan^{-1}\dfrac{k}{h}$ and altered in length by the factor $\sqrt{h^2 + k^2}$.

From this we get the following important rule.

If a vector is multiplied by a complex number of the form $h + jk$, there results a vector whose length is

$$\sqrt{(h^2 + k^2)} \times \text{(length of the original vector)}$$

and which is rotated relatively to the original vector through the angle $\tan^{-1} \dfrac{k}{h}$.

The operation is shown graphically in Fig. 31, where the vector $A(h + jk)$ is obtained by multiplying the vector A by the complex number $h + jk$.

From the figure it is also seen that the inclination of the new vector is that given on p. 50.

$$\text{For } \tan\theta = \frac{a_2}{a_1}$$

$$\tan\phi = \frac{k}{h}$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \,.\, \tan\phi} = \frac{\dfrac{a_2}{a_1} + \dfrac{k}{h}}{1 - \dfrac{a_2}{a_1} \cdot \dfrac{k}{h}} = \frac{a_2 h + a_1 k}{a_1 h - a_2 k}$$

The effect of multiplication or division by a complex number is more directly shown when the vector and complex number are expressed in their exponential form.

Exponential Form of Multiplication of a Vector by a Complex Number.—Let the vector $A = a_1' + a_2''$ be written in the form $A = a\varepsilon^{j\theta}$ (see p. 31), so that

$$a = \sqrt{a_1^2 + a_2^2}$$
$$a_1 = a \cos\theta$$
$$a_2 = a \sin\theta$$

where θ is the angle of inclination of the vector to the axis of reference.

Let the complex number be

$$\mathrm{K} = l + jm$$

where l and m are any simple numbers.

If α is an angle such that $\tan \alpha = \frac{m}{l}$ we may write the complex number in the form

$$\mathrm{K} = k\varepsilon^{j\alpha}$$

where $$k = \sqrt{l^2 + m^2}$$

K is, of course, not a vector, although it is now written in a form resembling that of a vector.

By multiplication of the exponential forms

$$\mathrm{K} \times \mathrm{A} = k\varepsilon^{j\alpha} \times a\varepsilon \quad = ka\varepsilon^{j(\theta + \alpha)} \quad . \quad . \quad . \quad . \quad (17)$$

This result represents a vector of the length ka and inclination to the axis $(\theta + \alpha)$.

The vector A has thus been increased in length by the factor k and rotated through an angle $\alpha = \tan^{-1}\frac{m}{l}$.

The parallel and normal components of the new vector may be seen by reconverting to the original form of notation.

$$\begin{aligned} ka\varepsilon j^{(\theta + \alpha)} &= ka\,\{\cos(\theta + \alpha) + j\sin(\theta + \alpha)\} \\ &= ka\,\{(\cos(\theta + \alpha))' + (\sin(\theta + \alpha))''\} \end{aligned}$$

The effect of division by the complex number is also at once evident from the exponential expressions.

Thus

$$\frac{a\varepsilon^{j\theta}}{k\varepsilon^{j\alpha}} = \frac{a}{k} \cdot \varepsilon^{j(\theta - \alpha)} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The resulting vector is thus seen to be numerically equal to the value of the original vector divided by the numerical value of the complex number, and in phase the resulting vector is rotated back through an angle α compared with the original vector A.

Examples taken from the Simple Electrical Circuit.—In a circuit containing both resistance and reactance, the total voltage applied to the circuit is the sum of the voltages overcoming resistance and reactance. These two voltages are at right angles in phase, having the values already found.

$$\text{Voltage overcoming resistance} = r\text{C}$$
$$\text{,,} \quad \text{,,} \quad \text{reactance} = -jx\text{C}$$

The total applied voltage may be written as the sum of these mutually perpendicular components, so that the applied voltage

$$\begin{aligned} &= \text{E} = r\text{C} - jx\text{C} \\ &= \text{C}(r - jx) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19) \end{aligned}$$

We know that the product of the current by the *impedance* of a circuit gives the voltage. The factor $r - jx$ is the symbolic expression for the impedance of the circuit. Evidently from its form, $r - jx$ is a complex number, and in consequence does not give the direct numerical value of the impedance. The expression gives the impedance in the form (resistance + reactance), with the addition of the symbol j which serves to separate these two components from one another.

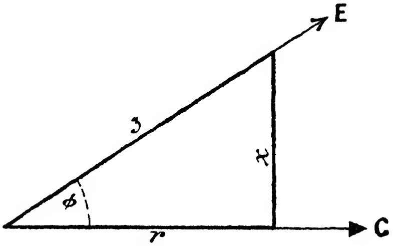

FIG. 32.—Triangle of impedance.

The components r and x will bear to one another and to the numerical value of the impedance ($= z$) the same relation as the sides of the right-angled triangle, having the sides Cr, Cx, Cz in which Cz is the hypotenuse. In this triangle, the angle between the side having r units of length and the side of length z will be the same angle as that between the current and voltage in the circuit, since this angle is $\tan^{-1}\frac{x}{r}$. It must, however, be kept clearly in mind that the triangle of impedance as shown in Fig. 32 is not a triangle of vectors, since the impedance, reactance, and resistance are measured in *ohms, i.e.* in units which are scalar, and have no sense of direction connected with them.

In a simple circuit it is often convenient to draw a triangle of impedance which is similar to, or even identical with, the triangle of voltages for the circuit, since the sides of the two triangles will be proportional to one another. In one case we have a triangle representing only *magnitudes* of scalar quantities, while in the other case the diagram is a diagram of vectors and shows *magnitudes and phase.*

Evidently from the impedance triangle we have for the numerical value of the impedance

$$z = \sqrt{r^2 + x^2}$$

In order to distinguish between the numerical and the complex, or symbolic, expression for the impedance, we may employ a similar convention to that used for the vectors, viz.

$$\mathrm{Z} = r - jx \text{ (complex form)}$$
$$z = \sqrt{r^2 + x^2} \text{ (numerical value)}$$

Thus symbolically

$$\mathrm{E} = \mathrm{CZ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and numerically

$$e = cz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If the axis of reference is taken along the vector of current, the expression for the applied voltage takes the form

$$\mathrm{E} = c'(r - jx) = rc' - xc''$$

In the more general case

$$\begin{aligned}\mathrm{E} &= \mathrm{C}(r - jx) = r\mathrm{C} - jx\mathrm{C} \\ &= r(c_1' + c_2'') - jx(c_1' + c_2'')\end{aligned}$$

Here the applied voltage is given in two terms, the first of which, $r(c_1' + c_2'')$, is evidently in phase with the current and represents the "energy" voltage of the circuit. The second component of the voltage, $-jx(c_1' + c_2'')$, is at right angles in phase to the current and represents the "idle" or "wattless" component of the applied voltage.

Thus when multiplying by the complex impedance we obtain these two components of the voltage separately

rC is the energy component of the voltage
$-jx$C is the idle ,, ,, ,,

In this case j has the signification that the terms in which it appears are "idle" or inoperative as regards the power of the circuit. This matter will be more clear after a perusal of Chapter VIII.

Effective Resistance.—It may be well to call attention here to an extended meaning which is frequently assigned to the voltage Cr. This voltage is always to be taken to be the "energy" voltage of the circuit, *i.e.* the component of the circuit voltage which is in phase with the current. In many cases, iron losses or induced voltages of rotation affect the phase relations of the circuit, so that the energy voltage is no longer numerically equal to the product of resistance and current. In these cases the voltage denoted in the vector diagram or in symbolic notation as the Cr voltage of the circuit is to be understood to indicate the total energy voltage, *i.e.* that voltage which when multiplied by the current will give the power of the circuit.

The energy voltage when divided by the current will give the *effective resistance* of the circuit, which will frequently be greater than the actual resistance. In all cases the effective resistance is such that when multiplied by the square of the current, the watts spent in the circuit are obtained.

A similar convention must make the full definition of the effective reactance x of the circuit to be "the reactance which, when multiplied by the current in the circuit, will give as a result the 'idle' voltage of the circuit."

Further Relations between Current and Voltage.—*Admittance, Conductance, Susceptance.*

The symbolic expression for the impedance of a circuit has been given already as

$$Z = r - jx$$

the numerical impedance being

$$z = \sqrt{r^2 + x^2}$$

By means of the impedance of a circuit we are able to find the voltage which must be applied to the circuit in order to maintain a known current.

If the voltage and impedance are known *numerically*, we can find the numerical value of the current by direct division of the voltage by the impedance.

$$c = \frac{e}{z}$$

If voltage and impedance are expressed *symbolically*, the relation becomes

$$C = \frac{E}{Z} = \frac{E}{r - jx} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Here we have a fraction of which the denominator is a complex number.

Division by Complex Number.—A fraction of the form just given (which will frequently be met with later) does not consist of the usual real and imaginary terms, and cannot have any meaning assigned to it until it has had the real and imaginary parts separated from one another, which can only be done by eliminating j from the denominator (*cf.* p. 47).

Rationalization.—In order to separate real and imaginary parts in the case of a fraction having a denominator of the form $a - jb$, we make use of the process of rationalization, *i.e.* we multiply both numerator and denominator by $a + jb$, thus obtaining $a^2 - j^2b^2 = a^2 + b^2$ as the denominator.

Similarly, if the denominator has the form $a + jb$, we obtain a like result by multiplication by $a - jb$.

In the present case in order to obtain separately the real and imaginary parts of

$$\frac{1}{r - jx}$$

we must proceed as follows:

$$\frac{1}{Z} = \frac{1}{r - jx} = \frac{r + jx}{(r - jx)(r + jx)} = \frac{r + jx}{r^2 + x^2}$$

since j^2 may be written equal to -1.

We thus have

$$\frac{1}{Z} = \frac{r}{r^2 + x^2} + j\frac{x}{r^2 + x^2} \quad . \quad . \quad . \quad . \quad (23)$$

that is, we have obtained an expression for $\frac{1}{\text{impedance}}$ in the form of a complex number (*cf.* pp. 28, 29).

$\frac{1}{Z}$, *i.e.* the inverse of the impedance of a circuit, is called the *admittance* of the circuit.

The symbol Y is usually employed for the symbolic expression for admittance, y being used to denote the numerical value. Hence

$$\text{admittance} = Y = \frac{r}{r^2 + x^2} + j\frac{x}{r^2 + x^2} \quad . \quad . \quad (24)$$

numerically

$$y = \frac{\sqrt{r^2 + x^2}}{r^2 + x^2} = \frac{1}{\sqrt{r^2 + x^2}} = \frac{1}{z} \quad . \quad (25)$$

The current in a circuit may be at once derived from the applied voltage and the admittance

$$\begin{aligned} C &= \text{voltage} \times \text{admittance} = E \times Y \\ &= E\left(\frac{r}{r^2 + x^2} + j\frac{x}{r^2 + x^2}\right) \end{aligned}$$

If the axis of reference is taken parallel to the voltage vector, we may write

$$\begin{aligned} C &= e'\left(\frac{r}{r^2+x^2} + j\frac{x}{r^2+x^2}\right) \\ &= \left(\frac{r}{r^2+x^2}e\right)' + \left(\frac{x}{r^2+x^2}e\right)'' \quad . \quad . \quad . \quad (26) \end{aligned}$$

In the general case, the current in the circuit will be given in the form

$$C = E\left(\frac{r}{r^2+x^2} + j\frac{x}{r^2+x^2}\right) \quad . \quad . \quad . \quad (27)$$

It is here important to notice that by multiplying the applied voltage of a circuit by the symbolic form of the admittance, we obtain the current as the sum of two components, viz.

One component

$$\frac{r}{r^2+x^2}E$$

which is in phase with the voltage, since the fraction $\frac{r}{r^2+x^2}$ is a simple number and does not change the phase of the voltage. This component is called the energy current of the circuit.

The other component is

$$j\frac{x}{r^2+x^2}E$$

which is 90° behind the voltage in phase, since its phase is that of E rotated by j through 90° in a counter-clockwise sense. This is called the idle current of the circuit.

Here again (as already on p. 56) we find j indicating the idle or wattless element in the circuit.

It is convenient to give special names to the two components of the admittance. In any circuit

$$\frac{r}{r^2+x^2}$$

is called the *Conductance*, and is usually indicated by the symbol g;

$$\frac{x}{r^2 + x^2}$$

is called the *Susceptance*, and is usually denoted by the letter b.

We thus obtain

$$\text{admittance} = \frac{1}{\text{impedance}} = (\text{conductance} + j\,\text{susceptance})$$
$$= g + jb$$

Also

gE = energy current of the circuit
jbE = idle " " "

where E is the *applied* voltage.

Trigonometrical Derivation of Conductance and Susceptance.—It is of interest to show that the current in a circuit may be obtained in terms of the conductance and susceptance from the ordinary trigonometrical form of the instantaneous value of the current.

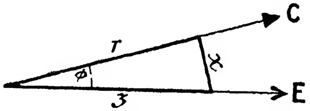

FIG. 33.—Relation between current and voltage.

In a circuit with resistance only, the instantaneous value of the current may be written

$$C_{\text{inst.}} = \frac{E}{r}\sin\theta$$

E being the maximum value of the applied voltage.

Where there is both reactance and resistance this becomes

$$C_{\text{inst.}} = \frac{E}{z}\sin(\theta - \phi)$$

ϕ being the angle of phase difference between current and voltage $= \tan^{-1}\frac{x}{r}$.

Hence

$$
\begin{aligned}
C_{\text{inst.}} &= \frac{E}{\sqrt{r^2+x^2}} \,.\, \sin(\theta-\phi) \\
&= \frac{E}{\sqrt{r^2+x^2}} \,.\, \sin\theta\cos\phi - \cos\theta\sin\phi \\
&= \frac{E}{\sqrt{r^2+x^2}} \cdot \left(\sin\theta\frac{r}{\sqrt{r^2+x^2}} - \cos\theta\frac{x}{\sqrt{r^2+x^2}}\right) \\
&= E\left(\sin\theta\ \frac{r}{r^2+x^2} - \cos\theta\cdot\frac{x}{r^2+x^2}\right)
\end{aligned}
$$

Since the sine and cosine of the varying angle θ go through identical values, and the only signification of these functions is to indicate a phase difference of 90° between the two terms of the current, we may indicate this characteristic of 90° phase difference by the symbol j, and write

$$C_{\text{inst.}} = E_{\text{inst.}}\left(\frac{r}{r^2+x^2} + j\frac{x}{r^2+x^2}\right)$$

whence $$C = E(g + jb)$$

The relations which we have now obtained, viz.

$$
\begin{aligned}
E &= C(r - jx) \text{ (equation (10))} \\
C &= E(g + jb) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)
\end{aligned}
$$

may be called the fundamental equations of the circuit. These equations are applied later to a number of examples in order to illustrate which equation should be used for various cases. Generally speaking, to find the voltage required for a given current we use (10), and to find the current due to a given applied voltage we use (28).

The close analogy between the conductance and susceptance as components of the admittance, and the resistance and reactance as components of the impedance, is well shown by constructing the diagrams for a circuit in which are resolved in one case the current and in the other case the voltage (see Figs. 34 and 35).

From the examples which follow, it will be seen that when a circuit contains impedances in series, the resultant impedance of the whole circuit is obtained by adding the individual impedances. If a circuit consists of branches connected in parallel, its impedance can only be calculated by first obtaining its total admittance by adding the admittances of the branches together. This method of solving problems is thus quite analogous to the treatment of continuous-current circuits in which resistances are in series or in parallel. The joint resistance of parallel circuits can only be obtained by first adding the conductivities of the branches to obtain the total conductivity.

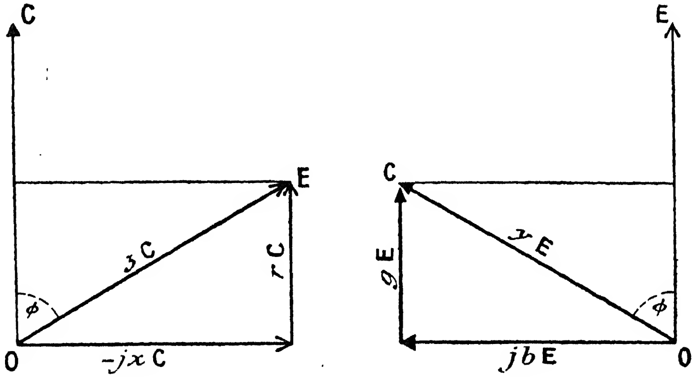

FIG. 34. FIG. 35.

Alternative methods of showing relation between current and voltage of a circuit.

Further Notes on Conductance and Susceptance.—Reference has already been made to the extended meaning which must often be given to *r*. This symbol usually denotes the resistance of a circuit, but it is also employed with the extended meaning of "the resistance which will give the total energy voltage applied to a circuit, when multiplied by the current in the circuit." This involves a similar extension of the meaning to be assigned to the terms "conductance" and "susceptance" when applied to circuits in which, owing to iron losses or other causes, the power given to the circuit is not equal to the numerical product (amperes)^2 × (ohms).

Our definitions of the "effective" conductance and susceptance of a circuit must run as follows:—

The effective conductance of a circuit is such that when multiplied by the voltage of the circuit the result is the energy current of the circuit, or when multiplied by the (voltage)^2 the result is the watts supplied to the circuit.

The effective susceptance of a circuit is such that when multiplied by the voltage of the circuit the result is the idle current of the circuit. When this susceptance is multiplied by the (voltage)^2 there results what is sometimes called the "wattless power" of the circuit.

We shall have frequent occasion later to speak of circuits in terms either of resistance and reactance or of conductance and susceptance. It is important to understand the connection between the reactance and susceptance of a circuit and between its resistance and conductance.

We have already shown that

$$\text{susceptance} = b = \frac{x}{r^2 + x^2}$$

from which it is evident that the susceptance is only equal to the reciprocal of the reactance when the resistance is zero. Similarly, the resistance and conductance are only reciprocal numbers when the reactance of the circuit is zero.

The use of the quantities and the analogy between them may be concisely expressed as follows:—

$$\begin{aligned} \text{Applied volts} \times \text{susceptance} &= \text{idle current} \\ \text{Current} \times \text{reactance} &= \text{idle voltage} \\ \text{Applied volts} \times \text{conductance} &= \text{energy current} \\ \text{Current} \times \text{resistance} &= \text{energy voltage} \end{aligned}$$

Special importance attaches to cases where one of the functions r, x, g, or b is constant in a circuit.

It is important also to notice that constant r does not involve constant g, except in a circuit of constant power-factor.

Similarly, if x is constant, b will not be constant (except in the special case of constant power-factor).

Constant r corresponds to an energy voltage in the circuit proportional to the current.

Constant g corresponds to an energy current proportional to the applied voltage.

Constant x corresponds to an idle voltage proportional to the current.

Constant b corresponds to an idle current proportional to the applied voltage.

These relations are of special importance in connection with locus diagrams (see Chap. IX.).

Circuits in Series and Parallel.—Let us consider the case of a circuit supplied with a voltage E, and composed of two impedances $Z_1 = r_1 - jx_1$ and $Z_2 = r_2 - jx_2$ connected in series.

The total impedance of the circuit will be

$$Z = Z_1 + Z_2 = r_1 + r_2 - j(x_1 + x_2)$$

The current in the circuit is given by the usual relation

$$C = \frac{E}{Z}$$

We can find the voltages spent in overcoming either of the impedances, Z_1 or Z_2, in terms of the total applied voltage and the individual resistances and reactances.

Let the voltage overcoming the impedance Z_1 be E_1, so that

$$E_1 = CZ_1$$

and similarly let

$$E_2 = CZ_2$$

Then

$$\begin{aligned} E &= E_1 + E_2 \\ &= C(Z_1 + Z_2) \\ &= \frac{E_1}{Z_1}(Z_1 + Z_2) \end{aligned}$$

Whence

$$E_1 = \frac{EZ_1}{Z_1 + Z_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and similarly

$$E_2 = \frac{EZ_2}{Z_1 + Z_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

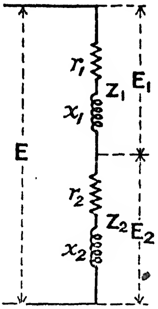

Fig. 36.—Impedances in series.

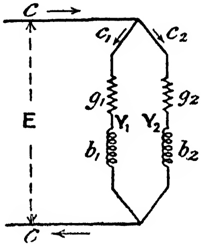

Fig. 37.—Admittances in parallel.

If the two circuits are in parallel, denoting their admittances by Y_1 and Y_2 we can obtain the branch currents in terms of the total current. We have the total admittance of the parallel circuit

$$Y = Y_1 + Y_2$$

Also the current in the branches is

$$C_1 = EY_1 \quad \text{and} \quad C_2 = EY_2$$

while the total current is

$$C = E(Y_1 + Y_2) = C_1 + C_2$$

From which we may eliminate E and obtain

$$C_1 = \frac{Y_1 C}{Y_1 + Y_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and

$$C_2 = \frac{Y_2 C}{Y_1 + Y_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

EXAMPLES IN ADMITTANCE AND IMPEDANCE.

Example 1.—A coil of wire wound upon a wooden core has an impedance of 10 ohms when the frequency is 50 $\sim$; its resistance is 1 ohm. What current will flow through it if it be put across a circuit of 100 volts 50 $\sim$, and what will be the power factor of this load?

$$c = \frac{e}{z} = \frac{100}{10} = 10 \text{ amps.}$$

Since in such a case (neglecting the small eddy currents which take place in the thickness of the wire) the only power developed is that due to resistance, we have

$$\cos \phi = \frac{r}{z} = \frac{1}{10} = 0{\cdot}1$$

Example 2.—What will the impedance of the above coil be if the frequency of the circuit be doubled?

We express Z as a complex number

$$Z = (r - jx)$$

and to find the value of x write

$$\begin{aligned} z &= \sqrt{r^2 + x^2} \\ 10 &= \sqrt{1 + x^2} \\ 100 &= x^2 + 1 \\ x &= \sqrt{99} = 3\sqrt{11} \end{aligned}$$

so

$$Z = (1 - j3\sqrt{11})$$

Now $x = 2\pi \sim l$ where l is the coefficient of self-induction of the circuit. Since there is no iron in the circuit

$$x \propto \sim$$

Thus if the $\sim$ be doubled

$$2x = 6\sqrt{11}$$

The resistance (if it be pure resistance) is not altered by the $\sim$.

So
$$\begin{aligned} Z &= r - j2x \\ &= 1 - j6\sqrt{11} \end{aligned}$$

and
$$z = \sqrt{(1)^2 + 396} = \sqrt{397} = 19{\cdot}9 \text{ ohms}$$

Corollary.—*When giving the impedance of a circuit it is necessary to state the corresponding frequency.*

Example 3.—Placing a core of iron in the above coil is found to double its impedance, and on fifty period circuits with a current of 10 amps. to increase the watts lost in the coil by 75 per cent. Give the iron loss, the apparent increase of resistance due to this loss, and show how much the reactance of the coil must have been increased by the presence of the iron.

The new impedance is $z = 20$ ohms.

Previously the watts lost with a current of 10 amps. were

$$c^2r = 10 \times 10 \times 1 = 100$$

Now they are 175. So if we call r_i the apparent increase of resistance due to the iron loss, we have

$$\begin{aligned} c^2(r + r_i) &= 175 \\ r + r_i &= \tfrac{175}{100} = 1{\cdot}75 \\ r_i &= 0{\cdot}75 \text{ ohms} \end{aligned}$$

Here r_i is not really a resistance, but a quantity having the dimensions of resistance, which, when multiplied by the square of the current, gives the iron loss in watts.

We can now find the new value of x, the reactance, as follows :—

$$\begin{aligned} z &= \sqrt{r^2 + x^2} \\ 20 &= \sqrt{(r + r_i)^2 + x^2} \\ &= \sqrt{(1{\cdot}75)^2 + x^2} \\ \text{whence} \qquad x &= \sqrt{400 - 3{\cdot}06} = \sqrt{396{\cdot}94} = 19{\cdot}9 \text{ ohms} \end{aligned}$$

Thus, by inserting the iron, the reactance has been increased from $3\sqrt{11}$ to 19·9 ohms on 50 ~ circuits.

Example 4.—Two coils are placed as a load upon a 50 ~ alternator whose P.D. is kept constant at 100 volts. The resistance of each coil is 2 ohms, and their coefficients of self-induction are 32 and 57 millihenrys respectively. State the current in the circuits and the power factor of this load (a) when the coils are placed in series, (b) when they are placed in parallel.

Let the reactance of the first coil be x_1 and that of the second coil be x_2.

$$\begin{aligned} \text{Then} \qquad x_1 &= 2\pi \sim l_1 = 2\pi \times 50 \times 0{\cdot}032 = 10 \text{ ohms} \\ x_2 &= 2\pi \sim l_2 = 2\pi \times 50 \times 0{\cdot}057 = 18 \quad ,, \\ \text{So} \qquad Z_1 &= (r_1 - jx_1) = (2 - j.10) \\ Z_2 &= (r_2 - jx_2) = (2 - j.18) \end{aligned}$$

(a). When they are in series their combined impedance is

$$Z = Z_1 + Z_2 = (4 - j.28)$$

of real value $z = \sqrt{16 + 784} = \sqrt{800} = 28{\cdot}2$ ohms

and the current is

$$c = \frac{e}{z} = \frac{100}{28{\cdot}2} = 3{\cdot}54 \text{ amps.}$$

$$\cos\phi = \frac{r_1 + r_2}{z} = \frac{4}{28{\cdot}2} = 0{\cdot}142$$

(*b*) When the coils are in parallel, the current through the first is

$$c_1 = \frac{e}{z_1}$$

Now $z_1 = \sqrt{104} = 10{\cdot}2$

and $z_2 = \sqrt{328} = 18{\cdot}12$

$$c_1 = \frac{100}{10{\cdot}2} = 9{\cdot}8 \text{ amps.}$$

$$c_2 = \frac{100}{18{\cdot}12} = 5{\cdot}52 \text{ amps.}$$

The current supplied by the alternator, however, is not the arithmetic sum of these, but their vector sum. It is best obtained by taking the joint admittance of the two coils, just as before we took their joint impedance.

$$\mathrm{Y}_1 = (g_1 + jb_1)$$

$$g_1 = \frac{r_1}{(z_1)^2} = \frac{2}{104} = 0{\cdot}0192$$

$$b_1 = \frac{x_1}{(z_1)^2} = \frac{10}{104} = 0{\cdot}096$$

Similarly

$$g_2 = \frac{r_2}{(z_2)^2} = \frac{2}{328} = 0{\cdot}0061$$

$$b_2 = \frac{x_2}{(z_2)^2} = \frac{18}{328} = {\cdot}0548$$

so

$$\begin{aligned} \mathrm{Y} &= \mathrm{Y}_1 + \mathrm{Y}_2 = g_1 + jb_1 + g_2 + jb_2 \\ &= 0{\cdot}0253 + j0{\cdot}1508 \\ y &= \sqrt{(0{\cdot}0253)^2 + (0{\cdot}1508)^2} = \sqrt{0{\cdot}0232} = 0{\cdot}1525 \\ c &= ey = 100 \times 0{\cdot}1525 \\ &= 15{\cdot}25 \text{ amps.} \end{aligned}$$

The vector sum of the two currents is in this case extremely near to the arithmetic sum; so near that the arithmetic sum would have been near enough. Indeed the difference is hardly within the limits of error in the calculation. But it is sufficient to show that differences not capable of measurement by scaling from a diagram may easily be calculated by vector algebra. The joint power factor is

$$\frac{g_1 + g_2}{y} = \frac{0{\cdot}0253}{0{\cdot}1525} = 0{\cdot}1665$$

Example 5.—A standard choking coil core has the dimensions shown in the accompanying sketch, Fig. 38.

It is arranged so that the air-gap at A is adjusted to 0·44″ per side.

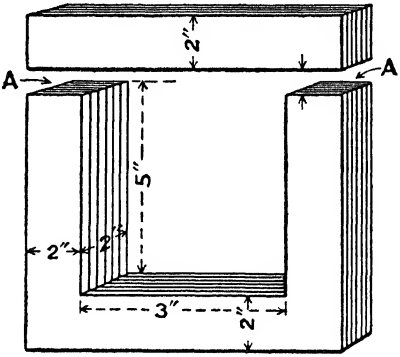

Fig. 38.—Dimensions of core of choking coil.

Give the relations between pressure and current for such a choker when wound with one layer per limb of No. 12 S.W.G. and worked on 50 ~ circuits.

The gross sectional area of the core is 4 square inches.

The net sectional area of the core may be taken as 3·6 square inches.

For clearness' sake we may first assume any density in the iron short of saturation, for such an assumption makes the meaning of the figures arrived at much more convincing.

Let us take a density of 50,000 lines per square inch.

Voltage and Flux.

Total maximum flux 3·6 × 50,000 = 180,000 C.G.S. lines.

Allowing for insulation, the available winding length is 4·5 inches on each side.

The diameter of No. 12 S.W.G. wire is 0·104, and when double cotton-covered it is best to allow for each turn 0·12 inches.

$$\text{Turns per layer (both limbs)} = \frac{2 \times 4{\cdot}5}{0{\cdot}12} = 75$$

Volts generated by the flux at a frequency of 50

$$e_2 = \frac{4{\cdot}44 \text{ max. flux} \times \text{turns} \times \sim}{10^8}$$
$$= \frac{4{\cdot}44 \times 18 \times 10^4 \times 75 \times 50}{10^8} = 30 \text{ volts}$$

Corresponding Ampere-turns.—Since the length of the path of a mean magnetic line is 24 inches, and for a density of 50,000 about 7 ampere-turns per inch are required in the iron, we get

$$\text{Ampere-turns for the iron} = 7 \times 24 = 168$$

The area of the air-gap is 4 square inches, and allowing for fringing may be taken at 4·5 square inches.

The density in the gap is thus (neglecting leakage)

$$\frac{180{,}000}{4 \cdot 5} = 40{,}000$$

Ampere-turns maximum for the two gaps

$$\begin{aligned} &= 0 \cdot 313 \times 0 \cdot 88 \times 40{,}000 \\ &= 1100 \text{ ampere-turns} \end{aligned}$$

Total for air and iron 1268 ampere-turns maximum = 900 R.M.S. ampere-turns
Length of mean turn of one layer = 9″ approx.
Resistance of 75 turns = 0·056 ohms

Value of Current.

Amperes magnetizing current R.M.S. $= \frac{900}{75} = 12$
Weight of iron in the core = 24 lbs.

Allowing 1·8 watts per lb. for iron losses

$$\text{watts} = 24 \times 1 \cdot 8 = 43 \cdot 2$$

$$\text{energy current} = \frac{43 \cdot 2}{30} = 1 \cdot 44 \text{ amps. R.M.S.}$$

So,

$$\begin{aligned} \text{total current} &= \sqrt{(\text{magnetizing current})^2 + (\text{energy current})^2} \\ &= \sqrt{(12)^2 + (1 \cdot 44)^2} = 12 \cdot 08 \text{ amps.} \end{aligned}$$

Now the E.M.F. denoted by E applied to the terminals of the choker may be resolved into two components—

(1) That which opposes the E.M.F. set up by the flux.

(2) That absorbed by the impedance of the windings.

Since we have neglected leakage flux in this case, the "impedance" becomes "resistance."

So we get the equation

$$\begin{aligned} E &= E_1 + Cz_1 \qquad (1) \\ &= E_1 + Cr_1 \text{ for this case} \qquad (2) \\ &= 30 + 12 \cdot 08 \times 0 \cdot 056 \end{aligned}$$

for E_1 is equal and opposite to E_2 (calculated above), and its value is therefore 30 volts, and the value of r_1 is 0·056 ohms.

This establishes the connection between E and C for one value of flux-density; and since a similar relationship will exist for every density up to the point of saturation of the iron, E_1 may be put in terms of C, as follows:—

$$E_1 = \frac{C}{g + jb}$$

Then, obtaining the values of g and b from the previous calculation:—

$$\begin{aligned} e_1 b &= 12 \text{ amps.} \\ b &= \tfrac{12}{30} = 0{\cdot}4 \text{ mho.} \\ e_1 g &= 1{\cdot}44 \text{ amps.} \\ g &= \frac{1{\cdot}44}{e_1} = \frac{1{\cdot}44}{30} = 0{\cdot}048 \text{ mho.} \end{aligned}$$

This gives us equation (2) in the form

$$\begin{aligned} E &= C\left(\frac{1}{g + jb} + r_1\right) \\ &= C\left(\frac{1}{0{\cdot}048 + j0{\cdot}4} + 0{\cdot}056\right) \quad . \; . \; . \; . \; . \quad (3) \end{aligned}$$

Even this may be improved upon by changing our admittance $(g + jb)$ into an impedance $(r - jx)$ as follows:—

We have

$$\begin{aligned} x &= \frac{b}{g^2 + b^2} = \frac{0{\cdot}4}{0{\cdot}162} = 2{\cdot}47 \text{ ohms.} \\ r &= \frac{g}{g^2 + b^2} = \frac{0{\cdot}048}{0{\cdot}162} = 0{\cdot}296 \text{ ohms.} \end{aligned}$$

Substituting in (2)

$$\begin{aligned} E &= \frac{C}{g + jb} + Cr_1 \\ &= C(r - jx) + Cr_1 \quad . \; . \; . \; . \; . \; . \; . \quad (4) \\ &= C\{(r + r_1) - jx\} = C\{0{\cdot}352 - j2{\cdot}47\} \end{aligned}$$

The meaning of r must be clearly kept in mind in this expression. It is not really a resistance; but it is a factor having the dimensions of resistance, such that when it is multiplied by the square of the current it gives the number of watts of iron losses corresponding to that current.

Similarly, the letter x, which often means reactance due to a leakage flux, in this case means reactance due to the flux existing in the iron of the core. With these distinctions well in mind, the calculations here given, and equations (1), (3), and (4), we solve almost any problem connected with an ordinary choking coil.

So long as the air-gap remains constant and the flux is not increased up to even partial saturation of the iron, the equations given will hold for the particular piece of apparatus. Let us work out one or two examples.

EXAMPLE 6.—Find the voltage drop across the choker when in series with an arc lamp requiring 10 amps.

Equation (4) $E = C(r - jx) + Cr_1$

$$E = C(0\cdot296 - j2\cdot47) + C(0\cdot056)$$

Hence $$e = 10\sqrt{(0\cdot296 + 0\cdot056)^2 + (2\cdot47)^2} = 25\cdot4$$

Notice how little r_1 affects the final result.

EXAMPLE 7.—Find the current and power factor when two such choking coils are put across a 50-volt 50 ~ mains in series.

Taking equation (4)

$$\begin{aligned} E &= 2C\{(r_0 - jx_0) + Cr_1\} \\ 50 &= 2C\{(r_0 + r_1) - jx_0\} \\ 25 &= C(0\cdot352 - j2\cdot47) \end{aligned}$$

So that 25 is the resultant of two lines at right angles whose individual lengths are proportional to 0·352 and 2·47 respectively.

$$c^2 \times (0\cdot352)^2 + c^2 \times (2\cdot47)^2 = 625$$

$$c^2 = \frac{625}{6\cdot46}$$

$$c = 9\cdot83 \text{ amps.}$$

The power factor is

$$\cos\phi = \frac{r_0 + r_1}{\sqrt{(r_0 + r_1)^2 + x_0^2}} = \frac{0\cdot352}{2\cdot54} = 0\cdot1385$$

Another way of arriving at this same result would be to express equation (4) so that the current might be directly calculated. This involves finding the actual value of the impedance of the circuit. Thus

$$z = \sqrt{(r_0 + r_1)^2 + x_0^2} = 2\cdot54$$

$$c = \frac{e}{2z} = \frac{50}{2 \times 2\cdot54} = 9\cdot83 \text{ amps.}$$

These examples introduce, and to some extent illustrate, the question of the relationships existing between main fluxes and leakage fluxes, and between admittances and impedances. To a further consideration of these points we shall now turn.

CHAPTER IV

SELF AND MUTUAL INDUCTION

Relationship of X, Y, and Z.—It has already been pointed out that where circuits in parallel are to be considered, it is as a rule more convenient to make use of admittances than of impedances, so that the calculation of the current takes the form

$$C = E_1Y_1 + E_1Y_2 + \ldots\ldots \qquad (33)$$

Similarly in series circuits it is more convenient to make use of impedances, so that the calculation of the voltage takes the form

$$E = C_1Z_1 + C_1Z_2 + \ldots\ldots \qquad (34)$$

In cases of series parallel grouping both forms may be of use or a combination of the two. Thus in the transformer we may get

$$E = (C_s + C_0)Z_1 + k^2C_sZ_2 + k^2C_sZ \text{ (equation (85), p. 109)}$$

where C_0 itself has a value expressed by

$$k^2C_s(Z_2 + Z)Y \text{ (see p. 110)}$$

Here we wish to make as clear as possible the connection between admittance and impedance, particularly as regards circuits containing iron.

Consider the simple case of a coil of *t* turns wound about a laminated iron core (Fig. 39). Such a coil will manifestly set up two fluxes, viz. F_m through the iron circuit, and flux like F_1 leaking round through various air paths. Fluxes through iron differ from fluxes mainly through air in that they are always accompanied by hysteresis and eddy currents in the iron itself. Both hysteresis and eddy currents have the same effect so far as the flux due to a given current is concerned, for they both tend to reduce the flux and to cause it to lag in phase behind the current which produces it. The flux through an iron path will always be greater than that through an air path of similar length and cross-section. The case of the fluxes in Fig. 39 may be illustrated by the vector diagram (Fig. 40), where C is the R.M.S. current which flows through the turns t.

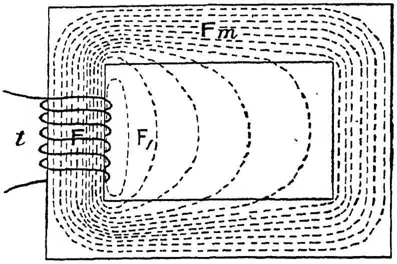

FIG. 39.—Flux distribution in a choking coil.

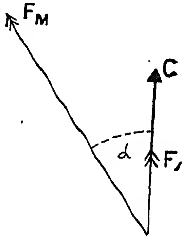

FIG. 40.—Iron circuit flux and current diagram.

F_1 is in phase with C, while F_m lags behind C in phase.

The angle α is called "the angle of hysteretic advance," or sometimes the iron loss angle.

It is evident that the maximum value of the flux F_1 is proportional to the current and turns, and may be expressed by

$$F_1 = h_1 t C \sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

(see equation (5), p. 42), where h_1 is constant and equal to the number of leakage lines set up by one ampere-turn, and t is the number of turns. The current is therefore given by

$$C = \frac{F_1}{\sqrt{2}h_1 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

h_1 is evidently a simple number.

We may write the relation between the flux F_m and the current in a similar manner

$$F_m = H_m t C \sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

In this case, H_m is a complex number because C and F_m are not in phase.

The flux F_m may be resolved along C and at right angles thereto.

This has been done in Fig. 41, and we get for the component in phase with C

$$F_a = \sqrt{2}m_1 t C$$

and for the component at right angles to C

$$F_b = j\sqrt{2}m_2 t C$$

where m_1 and m_2 are simple constants depending upon the dimensions of the iron path and the quality of iron, such that $m_1 = h_m \cos a$ and $m_2 = h_m \sin a$.

Thus in vector notation we have

$$F_m = \sqrt{2}tC(m_1 + jm_2) \quad . \quad . \quad . \quad . \quad . \quad (38)$$

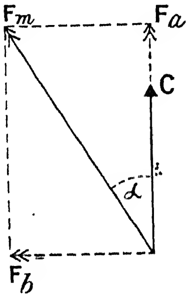

FIG. 41.—Iron circuit. Flux resolved with respect to current.

The constants h_1, m_1, m_2, might be chosen so as to include the constant t, but usually it is convenient to keep t separate.

Corresponding to $C = \frac{F_1}{\sqrt{2}h_1 t}$

we have
$$C = \frac{F_m}{\sqrt{2}t H_m} = \frac{F_m}{\sqrt{2}t(m_1 + jm_2)}$$ [1]

The total flux F will be composed of the fluxes in phase with C

$$F_a + F_1 = \sqrt{2}tC(m_1 + h_1)$$

and the flux at right angles in phase to C

$$F_b = \sqrt{2}tC(m_2)$$

Since m_1, m_2, and h_1 are all constants of the circuit and do not change so long as the flux density in the iron is not taken over a range wide enough to seriously change the value of the permeability of the iron path, it follows that we may write the total flux F linked with the coil as

$$F = F_m + F_1 = \lambda F_m \quad . \quad . \quad . \quad . \quad (39)$$

---

[1] This expression may be rationalized as follows :—

$$C = \frac{F_m}{\sqrt{2}t(m_1^2 + m_2^2)}(m_1 - jm_2)$$

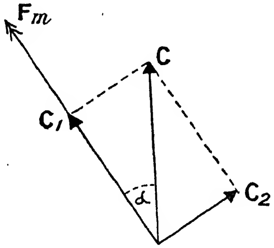

Fig. 42.—Current resolved with respect to flux in iron circuit.

In this form the current C is resolved along F_m and at right angles thereto, as shown in Fig. 42, where

$$C_1 = \frac{F_m}{\sqrt{2}t(m_1^2 + m_2^2)} m_1$$
$$C_2 = \frac{-jF_m}{\sqrt{2}t(m_1^2 + m_2^2)} m_2$$

F, F_m, F_1 being vectors and λ being a factor greater than unity, such that

$$\lambda = 1 + \frac{F_1}{F_m}$$

λ is then the Hopkinson leakage coefficient exactly as used in continuous current machine design. When dealing with continuous currents, however, there is no question of phase difference between F and F_m, as there is in this case. It thus appears that instead of being merely a number, as in the continuous current case, λ is here a complex quantity, shifting the phase of F_m as well as changing its magnitude.

As already stated

$$\begin{aligned} F &= F_m + F_1 = \lambda F_m \\ &= F_a + F_1 + F_b \end{aligned}$$

By substitution this becomes

$$F = \sqrt{2}tC(m_1 + h_1 + jm_2)$$

Or alternatively

$$F = \lambda F_m = \lambda(F_a + F_b) = \lambda\sqrt{2}tC(m_1 + jm_2) \quad . \quad (40)$$

Hence

$$m_1 + h_1 + jm_2 = \lambda(m_1 + jm_2) \quad . \quad . \quad . \quad (40a)$$

and

$$\begin{aligned} \lambda &= 1 + \frac{h_1}{m_1 + jm_2} \\ &= 1 + \frac{h_1(m_1 - jm_2)}{m_1^2 + m_2^2} \\ &= 1 + \frac{h_1}{\sqrt{m_1^2+m_2^2}}\left(\frac{m_1}{\sqrt{m_1^2+m_2^2}} - j\frac{m_2}{\sqrt{m_1^2+m_2^2}}\right) \quad (41) \end{aligned}$$

Note, however, that $\sqrt{m_1^2 + m_2^2}$ is the real value of the flux per ampere-turn produced in the iron path. We have called it h_m, being a simple number.

So

$$\lambda = 1 + \frac{h_1}{h_m}(\cos a - j \sin a) \quad . \quad . \quad . \quad (42)$$

$\cos a - j \sin a$ is sometimes written cis a. Consequently, λ may be expressed as $1 +$ (the ratio of the leakage lines produced per ampere-turn to the useful lines produced per ampere-turn) $\times$ an operator called cis a.

Note especially that h_1 and h_m are *not* vectors.

The above expression for λ is not very convenient for ordinary use. It is obvious, however, that if $a = 0$, *i.e.* if the iron losses are comparatively small, then $\lambda = 1 + \frac{h_1}{h_m}$ and is a pure number, as in the case of a dynamo.

We shall now consider the E.M.F.'s appropriate to these various fluxes.

The E.M.F. which the leakage flux F_1 will set up in the turns t is numerically

$$\begin{aligned} -e_x &= \sqrt{2}\pi f_1 t \sim 10^{-8} \\ &= \sqrt{2}\pi \cdot \times \sqrt{2} h_1 t^2 c \sim 10^{-8} \end{aligned}$$

from equation (35).

As a vector this voltage is 90° behind C (see Fig. 43).

Thus

$$-E_x = jC2\pi h_1 t^2 \sim 10^{-8}$$

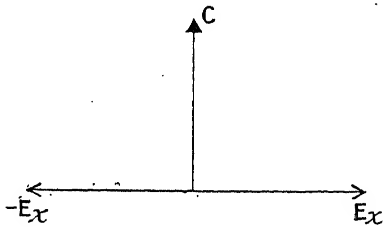

FIG. 43.—E.M.F.'s appropriate to a leakage flux.

The E.M.F. to be applied to the terminals of the coil to balance this is

$$E_x = -jC2\pi h_1 t^2 \sim 10^{-8}$$

Usually we denote the constant $\frac{2\pi h_1 t^2 \sim}{10^8}$ by the letter x, calling it the reactance of the circuit.

So
$$E_x = -jCx$$
or
$$C = \frac{E_x}{-jx}.$$

Rationalizing

$$C = j\frac{E_x}{x^2}x$$
$$= j\frac{E_x}{x}$$

$\frac{1}{x}$ may be written as another constant if we please (*cf.* p. 47), but usually the form $E_x = -jCx$ is quite convenient.

In the case of the flux F_m we shall consider independently

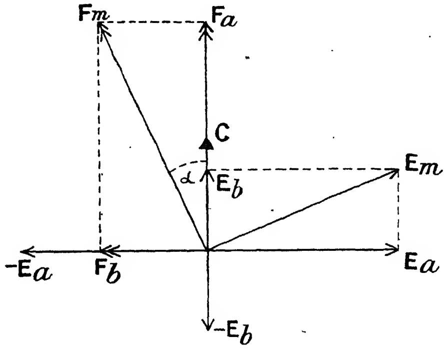

FIG. 44.—E.M.F.'s appropriate to iron path flux.

the E.M.F.'s set up by its two components (Fig. 44). Thus for the flux F_a we have the E.M.F.

$$-e_a = \sqrt{2}\pi f_a t \sim 10^{-8}$$
$$= \sqrt{2}\pi\sqrt{2}m_1 t^2 c \sim 10^{-8}$$

As a vector this is 90° behind C, thus

$$-E_a = jC2\pi m_1 t^2 \sim 10^{-8}$$

The E.M.F. to be applied to the terminals of the coil to balance this is

$$E_a = -jC2\pi m_1 t^2 \sim 10^{-8}$$

Here again the constants of the circuit

$$2\pi m_1 t^2 \sim 10^{-8}$$

may be combined into a single constant x_m.

So that $$x_m = 2\pi m_1 t^2 \sim 10^{-8} \quad . \quad . \quad . \quad (42a)$$

and $$E_a = -jCx_m$$

In the same way the flux F_b sets up a voltage

$$\begin{aligned} -e_b &= \sqrt{2}\pi f_b t \sim 10^{-8} \\ &= \sqrt{2}\pi \times \sqrt{2} m_2 t^2 c \sim 10^{-8} \end{aligned}$$

As a vector this is 90° behind F_b, or in antiphase as regards C; thus

$$-E_b = -C2\pi m_2 t^2 \sim 10^{-8}$$

so that the applied E.M.F. balancing this is

$$E_b = C2\pi m_2 t^2 \sim 10^{-8}$$

and is in phase with C. This being the case, the product of E_b and C represents power—that power, in fact, absorbed by hysteresis and eddy currents. For, if these be neglected, F_m is in phase with C, E_b disappears, and an E.M.F. like E_a remains at right angles to C, so that there is no power absorbed (see Chap. VIII.).

In choosing the letter to represent the constants

$$2\pi m_2 t^2 \sim 10^{-8}$$

we may have regard to the fact that these are concerned with an E.M.F. in phase with the current just like that due to resistance, and write

$$r_m = 2\pi m_2 t^2 \sim 10^{-8} \quad . \quad . \quad . \quad . \quad (42b)$$

Obviously the vector sum of E_a and E_b is an E.M.F. E_m at right angles to F_m and expressed by[1]

$$E_m = -jF_m\sqrt{2}\pi t \sim 10^{-8}$$

but

$$\begin{aligned} E_m &= E_b + E_a \\ &= C(r_m - jx_m) \\ &= CZ_m \text{ (say)} \end{aligned}$$

Now, if

$$E_m = C(r_m - jx_m)$$

then

$$C = \frac{E_m}{r_m - jx_m}$$

Rationalizing

$$C = \frac{E_m}{r_m^2 + x_m^2}(r_m + jx_m) \quad . \quad . \quad . \quad (43)$$

Here $\frac{1}{r_m^2 + x_m^2}(r_m + jx_m)$ is the admittance, Y, of the winding t in Fig. 39. Employing the symbols which we have previously adopted

$$Y = g + jb$$

Hence $g = \frac{r_m}{r_m^2 + x_m^2}$ and $b = \frac{x_m}{r_m^2 + x_m^2}$ (see p. 58)

r_m and x_m being defined as above, see equations (42*a*), (42*b*).

To sum up, then, when a coil of t turns surrounds an iron core and carries a current of R.M.S. value C it sets up a main flux F_m (having a numerical value f_m) and a leakage flux F_1 (with numerical value f_1). The E.M.F.'s to be applied to the terminals of the coil are

Cr to carry the current against the resistance of the coil r;

---

[1] In view of the possibility of the flux F_m passing through other coils on the core having turns more or less than t, it is convenient sometimes to separate t from the constants r_m and x_m and write

$$r_m = t^2 r_n$$
$$x_m = t^2 x_n$$

We shall then have

$$r_m - jx_m = t^2(r_n - jx_n)$$

and

$$Z_m = t^2 Z_n$$

where Z_n is written for $(r - jx_n)$.

$-jCx$ to counterbalance the E.M.F. set up by the flux F_1; and $C(r_m - jx_m)$ to counterbalance the E.M.F. set up by the flux F_m.

Thus the total E.M.F.

$$\begin{aligned} E &= C(r - jx) + C(r_m - jx_m) \\ &= CZ + CZ_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44) \end{aligned}$$

Relationship of Reactance and Coefficient of Self-Induction (x and L).—The coefficient of self-induction of a winding is the product of the turns of the winding and the number of lines which those turns embrace when carrying unit current. This product must be divided by 10^8 when the usual practical units are employed.

A coil carrying a current has a coefficient of self-induction L, which may be written as follows:—

$$L = \frac{\text{linkages of lines and turns}}{\text{current} \times 10^8} \text{ henrys}$$

Now, referring to Fig. 39, the flux per ampere-turn may be looked upon as including both F_m and F_1, or either separately. In physics it is usual to include both (Clerk Maxwell); but in electrical engineering it is more convenient and usual to separate them, and to include the *leakage fluxes only* in the coefficient of self-induction.

In the former case by definition we have

$$\begin{aligned} L &= \frac{(F_1 + F_m)t}{\sqrt{2}C \times 10^8} \\ &= \frac{\{\sqrt{2}h_1tC + \sqrt{2}tC(m_1 + jm_2)\}t}{\sqrt{2}C \times 10^8} \end{aligned}$$

The term $\sqrt{2}$ is introduced because the flux $\sqrt{2}h_1tC$ corresponds to the current $\sqrt{2}C$, since the maximum value of the flux is used.

Thus $$L = \frac{t^2(h_1 + m_1 + jm_2)}{10^8} \quad . \quad . \quad . \quad . \quad (45)$$

Now, $h_1 + m_1 + jm_2$ is the total flux *per ampere-turn* produced by and linked with the coil t.

We have written in equation ($40a$)

$$h_1 + m_1 + jm_2 = \lambda(m_1 + jm_2)$$

Hence
$$\mathrm{L} = t^2\lambda(m_1 + jm_2)10^{-8} \quad . \quad (46)$$

This symbol L is the coefficient of self-induction as used by Maxwell; which is obviously a complex quantity.

Again by definition we have for the second case, considering the flux F_1 only

$$l_1 = \frac{F_1 t}{\sqrt{2}C10^8} = h_1 t^2 10^{-8} \quad . \quad . \quad . \quad . \quad (47)$$

This symbol l_1 is the coefficient of self-induction as generally used by electrical engineers, and it is not a complex quantity, which accounts for its greater convenience.

We have previously (p. 78) written x the reactance

$$= \frac{2\pi h_1 t^2 \sim}{10^8}$$

hence
$$x = 2\pi \sim l_1$$

which is the familiar expression for the reactance of a circuit due to its leakage flux.

Relationship of the Two Coefficients of Self-Induction L and l_1.—Since the fluxes F_m and F_1 are not in phase, it follows that the relationship between l_1 and L cannot be a simple numeric; but must be a complex number. Thus by comparing equations (45) and (47)

$$\frac{\mathrm{L}}{l_1} = \frac{h_1 + m_1 + jm_2}{h_1} = 1 + \frac{m_1 + jm_2}{h_1}$$

which is a complex number.

We have already found in equation (42) that

$$\lambda = 1 + \frac{h_1}{h_m}(\cos a - j \sin a)$$

Note, however, that if there are no iron losses

$$\lambda = 1 + \frac{h_1}{h_m}$$

and

$$\frac{L}{l_1} = 1 + \frac{h_m}{h_1} = \lambda \ \ \lambda - 1 \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The relation between the two coefficients thus becomes a simple number when there are no iron losses, and is equal to the leakage factor of the circuit.

Iron Losses.—It is of interest also to note in Fig. 44 that the following ratios are equal :—

$$\frac{F_b}{F_a} = \frac{E_b}{E_a} = \frac{m_2}{m_1} = \tan a$$

but the quantity we have called

$$x_m = 2\pi m_1 t^2 10^{-8} \text{ (equation (42}a\text{))}$$

and

$$r_m = 2\pi m_2 t^2 10^{-8} \text{ (equation (42}b\text{))}$$

hence

$$\tan a = \frac{r_m}{x_m}$$

Now the iron losses, as already shown, are the product of c and e_b; *i.e.* of the current C and that component of the voltage (produced by the flux F_m) which is in phase with C.

But $\quad e_b = cr_m$ (see p. 80)

Hence the iron losses $= c^2 r_m$ . . . . (49)

If E_m represents as before the vector sum of E_b and E_a, we might write in a corresponding manner

Iron losses = product of E_m and that component of the current in phase with E_m

Now, this component is

$$E_m \frac{r_m}{r_m^2 + x_m^2}$$

or, as we have called it, $E_m g$.

Hence, as an alternative to the form given in equation (49), we may write

$$\text{iron losses} = e_m^2 g \quad . \quad . \quad . \quad . \quad . \qquad (50)$$

Again the angle in Fig. 44 by which E_m leads the current C is $(90° - a)$.

Hence

$$\cot a = \frac{F_a}{F_b} = \frac{E_a}{E_b} = \frac{m_1}{m_2} = \frac{x_m}{r_m} = \frac{b}{g}$$

Example.—A large choking coil has an iron core of the dimensions shown in the sketch. The number of turns on the limb is 20, and their resistance is 0·5 ohm; 705 ampere-turns are required to give a maximum density in the iron of 40,000 lines per square inch. Find the current corresponding to this flux density, and if the leakage factor is 1·02 find at 50 ~ the values of F_m, F_1, Z_m, Y; also the impedance Z of the coil and the values of flux in phase with the current, flux at right angles thereto, and the terminal volts. Also L and l_1.

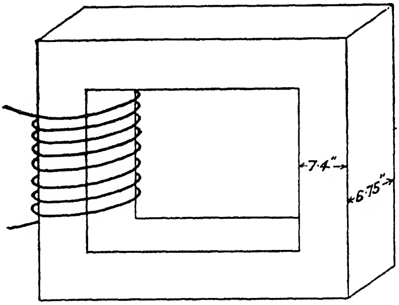

Fig. 45.—Dimensions of choking coil.

The length of mean magnetic line may be taken as 80", and the iron loss as 1000 watts.

Current.—Ampere-turns 705. Since this corresponds to the *maximum* flux density, it must be the maximum value of the ampere-turns.

$$\text{R.M.S. value} = \frac{705}{1{\cdot}41} = 500$$

$$\text{R.M.S. value of current} = \frac{500}{\text{turns}} = \frac{500}{20} = 25 \text{ amps.}$$

$$\begin{aligned}\text{Value of flux } F_m &= \text{section} \times \text{density} = 7{\cdot}4 \times 6{\cdot}75 \times 40{,}000\\ &= 2{,}000{,}000\\ &= 2 \times 10^6\end{aligned}$$

The Iron Loss Angle.

$$\begin{aligned}\text{Iron losses} &= 1000 \text{ watts} = c^2 r_m\\ r_m &= \tfrac{1000}{625} = 1{\cdot}6 \text{ ohms}\end{aligned}$$

$$\begin{aligned}\text{Value of} \quad e_m &= f_m\sqrt{2}\pi t \sim 10^{-8}\\ &= 2 \times 10^6 \times 4{\cdot}44 \times 20 \times 50 \times 10^{-8}\\ &= 88{\cdot}8 \text{ volts}\end{aligned}$$

$$\begin{aligned}\text{Now} \quad E_m &= C(r_m - jx_m)\\ \text{so} \quad e_m &= c\sqrt{r_m^2 + x_m^2}\\ 88{\cdot}8 &= \sqrt{40^2 + 25^2 x_m^2}\\ x_m &= 3{\cdot}17 \text{ ohms}\end{aligned}$$

hence (Fig. 44)

$$\tan a = \frac{r_m}{x_m} = \frac{1{\cdot}6}{3{\cdot}17}$$

$$a = 27^\circ$$

Value of $\frac{h_1}{h_m}$ *and Leakage Factor.*

We have

$$\sin a = 0{\cdot}45$$

$$\cos a = 0{\cdot}89$$

hence

$$\cos a - j \sin a = 0{\cdot}89 - j0{\cdot}45$$

From the equation (42) on p. 77, the leakage factor as a complex quantity is

$$\left(1 + \frac{h_1}{h_m} 0{\cdot}89\right) - j\frac{h_1}{h_m} 0{\cdot}45$$

The real value of this is given as

$$\lambda = 1{\cdot}02$$

Whence

$$\frac{h_1}{h_m} = \lambda - 1 = 0{\cdot}02$$

Value of f_1.

From equation (39) $$f_1 = 0{\cdot}02 f_m = 2 \times 10^6 \times \tfrac{2}{100}$$
$$= 4 \times 10^4 \text{ lines}$$

The Value of the Flux in Phase with the Current.

This is obviously $f_a + f_1$.

Now $$f_a = f_m \cos a$$
$$= 2 \times 10^6 \times 0{\cdot}89 = 1{\cdot}78 \times 10^6$$
and $$f_1 = 0{\cdot}04 \times 10^6$$
hence $$f_a + f_1 = 1{\cdot}82 \times 10^6$$

Flux at Right Angles to the Current.

$$f_b = f_m \sin a$$
$$= 2 \times 10^6 \times 0{\cdot}45 = 0{\cdot}9 \times 10^6$$

Value of Z$_m$ *and* Y.

$$Z_m = r_m - jx_m$$
$$= 1{\cdot}6 - j3{\cdot}17$$
$$z_m = 3{\cdot}55 \text{ ohms}$$
$$g = \frac{r_m}{r_m^2 + x_m^2} = \frac{1{\cdot}6}{12{\cdot}56} = 0{\cdot}127$$
$$b = \frac{x_m}{r_m^2 + x_m^2} = \frac{3{\cdot}17}{12{\cdot}56} = 0{\cdot}25$$
$$Y = g + jb = 0{\cdot}127 + j0{\cdot}25$$
$$y = 0{\cdot}28 \text{ mho}$$

Value of the Leakage Reactance x and of l_1.

$$l_1 = \frac{f_1 t}{\sqrt{2}c10^8} = \frac{4 \times 10^4 \times 20}{\sqrt{2} \times 25 \times 10^8}$$
$$= 0{\cdot}000225 \text{ henry}$$
$$2\pi \sim l_1 = x = 2\pi \times 50 \times 0{\cdot}000225$$
$$= 0{\cdot}0708 \text{ ohm}$$

Value of Z.

$$Z = r - jx = 0{\cdot}5 - j0{\cdot}07$$
$$z = 0{\cdot}502 \text{ ohm}$$

Value of Maxwell's Coefficient of Self-induction L.

$$L = \frac{(F_1 + F_m)t}{\sqrt{2}c \times 10^8} = \frac{t}{\sqrt{2}c \times 10^8}\{(F_1 + F_a) + F_b\}$$

$$= \frac{20}{\sqrt{2} \times 25 \times 10^8}(1{\cdot}82 \times 10^6 + j0{\cdot}9 \times 10^6)$$

$$= \frac{2}{\sqrt{2} \times 250}(1{\cdot}82 + j0{\cdot}9) = 0{\cdot}00564(1{\cdot}82 + j0{\cdot}9)$$

The numerical value of this is 0·0113.

Terminal Voltage.—The values of L and l_1, x, Z_m are very conveniently checked in the calculation of the terminal voltage. For it is clear

(1) That the terminal voltage must be equal to the vector sum of the voltage due to the fluxes F_m and F_1 and that required by the resistance of the coil. Thus

$$E = -j\sqrt{2}\pi t \sim \overline{F_m + F_1} \times 10^{-8} + Cr$$

(2) That E must be the vector sum of the voltage due to self-induction as expressed by Maxwell's coefficient L, and of that required by the resistance of the coil; or

$$E = -j2\pi \sim LC + Cr$$

(3) That E must likewise be the vector sum of the voltage required by the main flux F_m (called sometimes the back E.M.F.), of the reactance voltage, and of the resistance voltage; or

$$E = -j\frac{\sqrt{2}\pi t \sim}{10^8}F_m - jCx + Cr$$

Thus we have the identities with their values

$$(1).\quad E = -j\frac{\sqrt{2}\pi t \sim}{10^8}(F_m + F_1) + Cr$$

$$= -j\,\frac{\sqrt{2}\pi t \sim}{10^8}\,(F_a + F_1 + F_b) + Cr$$

or with C as axis of reference

$$\begin{aligned}E &= -4{\cdot}44 \times 20 \times 50 \times 10^{-8}\{(f_a + f_1)'' - f_b'\} + cr' \\ &= -4440 \times 10^{-8}\{(1{\cdot}82 \times 10^6)'' - (0{\cdot}9 \times 10^6)'\} + (25 \times 0{\cdot}5)'\end{aligned}$$

Note here (see Figs. 40 and 44) that F_a, F_1, and C are in phase, hence the E.M.F. due to the former is at right angles to that due to F_b and to Cr.

$$\begin{aligned}\text{Thus } e &= \sqrt{\{(44{\cdot}4 \times 0{\cdot}9) + (25 \times 0{\cdot}5)\}^2 + (44{\cdot}4 \times 1{\cdot}82)^2} \\ &= 96 \text{ volts}\end{aligned}$$

(2) $$\begin{aligned}E &= -j2\pi \sim LC + Cr \,(\text{note that } -j2\pi \sim LC \text{ is } 90^\circ \text{ ahead of C}) \\ &= -j6{\cdot}28 \times 50 \times 0{\cdot}00564(1{\cdot}82 + j0{\cdot}9)25' + 25 \times 0{\cdot}05' \\ E &= 40' + 12{\cdot}5' - 80'' \\ e &= 96 \text{ volts}\end{aligned}$$

(3) $$\begin{aligned}E &= -j\sqrt{2}\pi t \sim F_m - jCx + Cr \\ &= -j\sqrt{2}\pi t \sim F_a + \sqrt{2}\pi t \sim F_b - jCx + Cr\end{aligned}$$

or with C as axis of reference

$$E = -\sqrt{2}\pi t \sim f_a'' + \sqrt{2}\pi t \sim f_b' - cx'' + cr'$$

Taking together the second and last terms we have

$$44{\cdot}4 \times 0{\cdot}9 + 12{\cdot}5 = 52{\cdot}5$$

And the first and third terms are

$$\frac{-4{\cdot}44 \times 20 \times 50 \times 1{\cdot}78 \times 10^6}{10^8} - 25 \times 0{\cdot}07 = -80$$

Hence $$\begin{aligned}E &= 52{\cdot}5' - 80'' \\ e &= \sqrt{(52{\cdot}5)^2 + (80)^2} \\ &= 96 \text{ volts}\end{aligned}$$

Mutual Induction, Leakage, and Dispersion Coefficients.[1]—If upon the iron core in Fig. 39 we wind two coils having turns t_1 and t_2 respectively as in Fig. 46, we shall not (when

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[1] The student who finds difficulty in following the remaining portion of this chapter may omit it, as the considerations introduced are not essential to the understanding of the remaining chapters.

t_1 is connected to the source of supply, and coil t_2 is kept open) alter in any way the distribution of the fluxes F_m and F_1. But the flux F_m is now linked with the coil t_2, and in virtue of this fact, and because it is varying sinusoidally, it

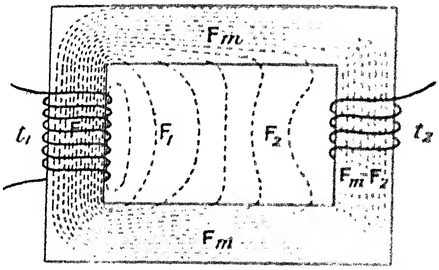

FIG. 46.—Distribution of fluxes in the transformer under load.

sets up in the turns t_2 an E.M.F. E_2. The coils t_1 and t_2 need not necessarily have the same number of turns.

Let $\quad t_1 = kt_2 \quad$ so that $\quad \dfrac{t_1}{t_2} = k$

Evidently $\quad E_2 = j\sqrt{2}\pi \sim t_2 F_m 10^{-8}$

but if $m_1 + jm_2$ as before is the maximum flux per ampere-turn in the iron (see p. 75), then

$$F_m = \sqrt{2}C_1(m_1 + jm_2)t_1$$

where C_1 is the R.M.S. current in t_1.

Substituting this value in the former equation

$$E_2 = j2\pi \sim t_1 t_2(m_1 + jm_2)C_1 10^{-8} \quad . \quad . \quad (51)$$

If now the coil t_2 carries a current C_2 it will set up in the coil t_1 an E.M.F.

$$E_1 = j2\pi \sim t_1 t_2(m_1 + jm_2)C_2 10^{-8} \quad . \quad . \quad . \quad (52)$$

The part $t_1 t_2(m_1 + jm_2)10^{-8}$ occurring in both these equations was called by Maxwell the *coefficient of mutual* *induction*, and is usually denoted by M. It accords exactly with the coefficient of self-induction L previously referred to, and may consequently be defined as "the linkage of magnetic lines by one coil due to unit current flowing in the other divided by 10^8."

Thus $\quad E_2 = j2\pi \sim MC_1$
and $\quad E_1 = j2\pi \sim MC_2$

And just as we have written

$$2\pi \sim l_1 = \text{reactance}$$

so we might appropriately write

$$2\pi \sim M = \text{mutual impedance}$$

impedance being written instead of reactance because M is a complex number containing $m_1 + jm_2$.

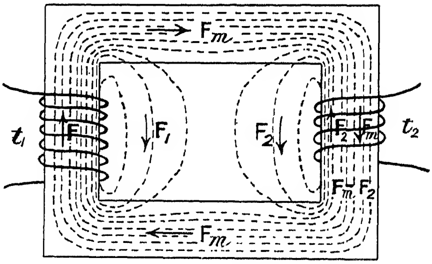

FIG. 47.—Resolution of fluxes in the transformer.

When the coil t_2 forms part of a closed circuit so that it carries a current, it also may cause extra leakage flux F_2 across paths similar to those traversed by F_1. This leakage flux will be practically in phase with C_2 and may therefore be taken into account by a coefficient of self-induction like L or l_1. We will call Maxwell's coefficient of self-induction for the coil t_2, L′, and the *Electrical Engineer's* coefficient l_2.

L′ then corresponds to L, and its value is given by

$$L' = t_2{}^2\lambda_2(m_1 + jm_2)10^{-8} \quad . \quad . \quad . \quad (53)$$

where λ_2 is the leakage factor for coil t_2 (*cf.* equation (46)). Similarly, by equation (47)

$$l_2 = t_2^2 h_2 10^{-8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where h_2 is the number of leakage lines per ampere-turn of coil t_2, and the reactance of the coil t_2 is

$$x_2 = 2\pi \sim l_2$$

which is again the usual form of the reactance due to the secondary leakage flux.

NOTE TO FIGS. 46 AND 47.—It may be well to compare Fig. 46 with Fig. 47, which represents the same conditions, but in a slightly different way. In Fig. 46 all the lines of flux are shown as being enclosed by t_1, while some complete their path through the core and some through the air. Physically, this is probably the best method of representation, but it suffers from the disadvantage that it does not distinguish between the fluxes F_1 and F_2, and does not make it apparent why these should not be in phase with one another. In Fig. 47 the loop of flux linked with the coil t_2 alone, marked F_2, is more evidently due to the ampere-turns of this winding, being shown as distinct from F_1. The two fluxes F_2, F_m do not, of course, actually exist side by side within the coil t_2; it is the vector sum of these fluxes which is actually in this part of the core.

Relationship of M, L, and L'.

Since $\quad M = t_1 t_2(m_1 + jm_2)10^{-8}$ (p. 90)

and $\quad L = t_1^2\lambda_1(m_1 + jm_2)10^{-8}$ (equation (46))

it follows that

$$\frac{L}{M} = k\lambda_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Similarly

$$\frac{L'}{M} = \frac{1}{k}\lambda_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Relationship of M, l_2, and l_1.

Since $\quad h_1 + m_1 + jm_2 = \lambda_1(m_1 + jm_2)$ (equation (40*a*))

we get $\quad h_1 = (\lambda_1 - 1)(m_1 + jm_2)$

and similarly $\quad h_2 = (\lambda_2 - 1)(m_1 + jm_2)$

Whence

$$\frac{l_1}{M} = k(\lambda_1 - 1) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

$$\frac{l_2}{M} = \frac{1}{k}(\lambda_2 - 1) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (58)$$

Relationship of l_1, L, l_2, L'.—It directly follows from the foregoing that

$$\frac{l_1}{L} = \frac{\lambda_1 - 1}{\lambda_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (59)$$

and

$$\frac{l_2}{L'} = \frac{\lambda_2 - 1}{\lambda_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (60)$$

We now have clearly on Maxwell's system the following E.M.F.'s affecting coil t_1:—

(1) The applied E.M.F. E.

(2) The E.M.F. required by the primary resistance $C_1 r_1$.

(3) The E.M.F. required by primary self-induction $-j2\pi \sim LC_1$.

(4) The E.M.F. required by secondary mutual induction $-j2\pi \sim MC_2$.

By this system, then, the assumptions made are, that the primary current-turns set up a flux through t_2 which in the case of t_1 is included in L; and that the secondary current-turns set up a corresponding (and independent) flux included for the coil t_2 in L′, but appearing in t_1 in the factor M.

General Equation for the Applied E.M.F.—By equating the voltage applied to a coil to the pressures opposing this voltage, we obtain what may be called the general equation for the circuit.

The equation for the coil t_1 in Fig. 46 or 47 is then

$$E = -j2\pi \sim LC_1 - j2\pi \sim MC_2 + C_1 r_1 \quad . \qquad (61)$$

From exactly similar reasoning, for the secondary circuit t_2 we have

$$-j2\pi \sim L'C_2 - j2\pi \sim MC_1 + C_2 r_2 = 0 \quad . \qquad (62)$$

where r_2 and L′ have reference to the *whole* circuit, *i.e.* coil and external circuit (if any).

Substituting in the latter equation for M, $\frac{kL'}{\lambda_2}$, we get

$$-j2\pi \sim L'C_2 - j2\pi \sim \frac{kL'}{\lambda_2}C_1 + C_2r_2 = 0$$

whence
$$C_2 = \frac{j2\pi \sim \frac{kL'}{\lambda_2}}{-j2\pi \sim L' + r_2}C_1 \quad . \quad . \quad . \quad . \quad (63)$$

Substituting this in the equation (61) for coil t_1 and writing $M = \frac{L}{k\lambda_1}$

$$E = -j2\pi \sim LC_1 - j2\pi \sim \frac{L}{k\lambda_1} \cdot \frac{j2\pi \sim \frac{kL'}{\lambda_2}C_1}{-j2\pi \sim L' + r_2} + C_1r_1 \quad . \quad (64)$$

Derivation of Dispersion Coefficient.—Two special cases of this equation are deserving of attention. The first is that which obtains when r_2 is made infinitely great, *i.e.* when the secondary is on open circuit and $C_2 = 0$. We get then

$$E = -j2\pi \sim LC_1 + C_1r_1 \quad . \quad . \quad . \quad . \quad (65)$$

Or if C_1r_1 is very small, as usually it is

$$E = -j2\pi \sim LC_1 \quad . \quad . \quad . \quad . \quad . \quad (66)$$

a form which is already familiar to us as expressing the relation between the applied voltage and current in a purely reactive circuit.

The second case of importance is that which occurs when the secondary is short-circuited, so that r_2 is very small and L′ contains no term referring to the external circuit. Then

$$E = -j2\pi \sim LC_1 + j2\pi \sim \frac{L}{\lambda_1\lambda_2}C_1 + C_1r_1$$

or
$$E = -j2\pi \sim LC_1\left(1 - \frac{1}{\lambda_1\lambda_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

if C_1r_1 be again considered as very small.

Now, if the applied potential difference E is kept constant, and C_0 be the current with t_2 on open circuit, C_x that with t_2 on short circuit, then evidently

$$j2\pi \sim LC_0 = j2\pi \sim LC_x\left(1 - \frac{1}{\lambda_1\lambda_2}\right)$$

and

$$\frac{C_0}{C_x} = 1 - \frac{1}{\lambda_1\lambda_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

This relationship is sometimes called σ, *the dispersion coefficient.* If there are no iron losses, then λ_1 and λ_2 are simple numbers. Hence C_0 and C are then in phase, and both at right angles to the E.M.F. producing them.

Sometimes,[1] however, $\frac{C_0}{C_x - C_0}$ is called the dispersion coefficient. We shall call this coefficient ν, so that

$$\nu = -\lambda_1\lambda_2$$

Throughout this book we shall attach the former meaning to the dispersion coefficient, *i.e.*

$$\sigma = 1 - \frac{1}{\lambda_1\lambda_2}$$

Connection between Equations employing L and *l*.—The foregoing equations and relationships have been developed mainly to link up the historic equations of Maxwell and his many followers with those more recent and convenient forms established by Hopkinson, Steinmetz, and others. We shall complete the analysis by developing from Maxwell's equations those of Hopkinson and Steinmetz. It will be noticed that throughout we have not differentiated between resistance due to the coil t_2 and resistance which might be, and usually is, externally inserted in the circuit of t_2. The same remark applies to L′, which includes any external self-induction, here, however, taken as depending solely upon M and λ_2. In practice the resistance and reactance external to a transformer secondary are variable and in no way dependent upon M. It is desirable, therefore, to separate them out. These facts account for the Maxwell equations being chiefly of use for transformers upon open or short circuit, or, at the most, on non-inductive load. For induction motors, on the other hand, they may be of great use. In order to separate external resistance and reactance from the corresponding internal quantities it is necessary also to separate leakage flux from main flux, *i.e.* to break up L′ and L into their constituents containing l_2 and l_1 and M. Then it is easy to divide l_2 into that part including the leakage flux in the apparatus itself, and that part which appertains to any external load connected thereto. For the present we shall separate out l_2 and l_1 without any reference to an external load; in a subsequent chapter we shall deal fully with the conditions involving the latter.

---

[1] *Cf.* p. 133.

The equation to the secondary coil may be written

$$-j2\pi \sim L'C_2 + r_2C_2 = j2\pi \sim MC_1 \text{ (equation (62))}$$

Since $L' = \frac{M}{k}\lambda_2$, this may be written

$$-j2\pi \sim \frac{M}{k}\lambda_2C_2 + C_2r_2 = j2\pi \sim MC_1$$

Now let $\frac{C_2}{k} = -C_s$. So that C_s is a current opposite in sense to C_2 and reduced in magnitude by the ratio of the turns, k, so that

$$C_2t_2 = -C_st_1$$

Then the equation becomes

$$j2\pi \sim M\lambda_2C_s - j2\pi \sim MC_1 = -C_2r_2$$

But $\lambda_2 = \frac{kl_2}{M} + 1$ (see equation (58)).

$$\text{So} \quad -j2\pi \sim M(C_s - C_1) = C_2r_2 - j2\pi \sim C_2l_2 \quad . \quad (69)$$

Obviously the right-hand side of this equation expresses the total E.M.F. existing in the secondary circuit. The left-hand side is the total E.M.F. generated in the secondary circuit by the flux coming through it; we may write then

$$E_2 = j2\pi \sim M(C_1 - C_s) = C_2r_2 - j2\pi \sim C_2l_2 \quad . \quad (70)$$

The meaning of this equation is that the total secondary E.M.F. is the product of 2π times the coefficient of mutual induction into the vector difference between the primary current (C_1), and the secondary current $\times \frac{t_1}{t_2}$, *i.e.* reduced to terms of the primary circuit. This is the basis of the Hopkinson and Steinmetz equations.

By a precisely similar substitution the equation for the primary circuit may be written

$$\begin{aligned} E &= -j2\pi \sim Mk(C_1 - C_s) + C_1 r_1 - j2\pi \sim l_1 C_1 \\ &= -kE_2 + C_1 r_1 - j2\pi \sim l_1 C_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (71) \end{aligned}$$

With the equations in this form it is easy to deal with external load, as will be shown in the next chapter.

Coefficient of Mutual Induction, Z_m and Y.—We called $Z_m (= r_m - jx_m)$ the main impedance in the case of the choking coil (Chap. IV., p. 80).

$$\begin{aligned} \text{Now,} \quad r_m - jx_m &= 2\pi \sim t_1^2 10^{-8}(m_2 - jm_1) \, (42a \text{ and } 42b) \\ &= -j2\pi \sim t_1 t_2 k 10^{-8}(m_1 + jm_2) \\ &= -j2\pi \sim kM \end{aligned}$$

Hence in the secondary circuit, instead of equation (69)

$$j2\pi \sim M(C_1 - C_s) = C_2 r_2 - j2\pi \sim C_2 l_2$$

we might write

$$\begin{aligned} -\frac{1}{k} Z_m(C_1 - C_s) &= C_2 r_2 - j2\pi \sim C_2 l_2 \\ &= C_2(r_2 - jx_2) \quad . \quad . \quad . \quad . \quad (72) \end{aligned}$$

And similarly equation (71) for the primary circuit will read

$$\begin{aligned} E &= Z_m(C_1 - C_s) + C_1 r_1 - j2\pi \sim l_1 C_1 \\ &= Z_m(C_1 - C_s) + C_1(r_1 - jx_1) \quad . \quad . \quad (73) \end{aligned}$$

$$\begin{aligned} \text{so} \qquad 2\pi \sim kM &= x_m + jr_m \\ &= j(r_m - jx_m) = jZ_m \end{aligned}$$

Again, the admittance $Y = \frac{1}{Z_m}$ (see p. 81)

$$\text{hence} \qquad -\frac{1}{j2\pi \sim kM} = Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Connection between Reactance, Mutual Impedance, and Leakage Factor.—From the expression $h_1 + m_1 + jm_2 = \lambda_1(m_1 + jm_2)$ it is very easy to deduce the following relationships :—

$$r_m - j(x_m + x_1) = \lambda_1 Z_m \quad . \quad . \quad . \quad . \quad . \quad (75)$$

and

$$r_m - j(x_m + k^2x_2) = \lambda_2 Z_m \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Relationship of $x_1 - x_2$, Z_m, and σ.—When the coil t_2 is short-circuited and the voltage E is applied to t_1, we have from equation (72) to the secondary circuit on p. 97

$$Z_m(C_1 - C_s) = k^2C_s(r_2 - jx_2) = k^2C_sZ_2$$

whence

$$\frac{C_1}{C_s} = \frac{k^2Z_2 + Z_m}{Z_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

If again r_2 is small as compared with x_2, then evidently

$$\frac{C_1}{C_s} = \lambda_2 \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad (78)$$

a most important practical relationship.

Now, writing down the equation (73) to the primary circuit

$$E = Z_m(C_1 - C_s) + C_1(r_1 - jx_1)$$

Then, neglecting r_1 (as before) and writing C_x for the corresponding value of the primary current, we get

Value of the primary current with secondary on short circuit

$$= c_x = \frac{c}{\dfrac{k^2x_2}{\lambda_2} + x_1}$$

Similarly with the coil t_2 on open circuit and the same applied E.M.F., when the iron losses are neglected the current is given by

$$c_0 = \frac{e}{x_m + x_1} = \frac{e}{\lambda_1 x_m}$$

So the relationship

$$\frac{\text{open circuit current}}{\text{short circuit current}} = \frac{C_0}{C_x} = \frac{k^2x_2 + \lambda_2x_1}{\lambda_1\lambda_2x_m}$$

or since $\frac{1}{x_m} = b =$ the susceptance

$$\frac{C_0}{C_x} = b\left(\frac{k^2x_2 + \lambda_2x_1}{\lambda_1\lambda_2}\right) = \sigma \quad . \quad . \quad . \quad . \quad (79)$$

Now
$$\frac{k^2x_2 + \lambda_2x_1}{\lambda_1\lambda_2x_m} = \frac{(\lambda_2 - 1)x_m + \lambda_2(\lambda_1 - 1)x_m}{\lambda_1\lambda_2x_m}$$
$$= 1 - \frac{1}{\lambda_1\lambda_2} = \sigma$$

Compare with this equation (68), p. 95.

In the succeeding chapter we shall develop the equations to the transformer without reference to Maxwell's form of the self and mutual induction.

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CHAPTER V

THE TRANSFORMER

THE adoption of alternating currents for power and lighting distribution has been brought about chiefly on account of the ease with which transformation can be effected from high voltage and small current to low voltage and large current, whereby a great saving can be obtained in the amount of copper required for transmission, whilst safe distribution is still quite easy. For this reason transformation is an essential part of almost every alternate-current scheme, and the transformer a most important piece of apparatus. Besides this, however, it so happens that every form of motor of the induction type, single or polyphase, with commutator or without, involves the principle of the transformer, so that in understanding and dealing with this apparatus thoroughly, we master at once the questions underlying three-quarters of the problems in alternate-current working.

Essentially, the transformer consists of a magnetic circuit interlinked with two electric circuits, the primary and the secondary. Pressure is applied to the terminals of the primary, and a magnetic flux results in the magnetic circuit. This in turn sets up a pressure in the secondary circuit, so that from the latter power may be taken. When a current is taken from the secondary, the ampere-turns due to the secondary windings react upon the primary through the medium of the magnetic circuit in such a manner as to call for ampere-turns in the primary circuit to balance them; so that a change of current supplied to the primary circuit ensues which exactly represents the change in the secondary current.

Such, put very roughly and briefly, are the elementary principles of the transformer set down here for the purpose of showing from what point of view we now approach the matter.

Magnetic Fluxes.—As we look into the transformer more closely, it is apparent that the ampere-turns of the primary and secondary coils can set up fluxes in other paths besides that formed by the iron circuit which unites them; these other fluxes are commonly called "leakage" fluxes or "self-inductions," and are taken account of in various ways. The method we adopt (which agrees with that used by Steinmetz), is explained thus:—

Fig. 48 represents diagrammatically a transformer, I being the primary coil, and II the secondary coil, united by the laminated iron core.

When the R.M.S. pressure E of value *e* is applied to the primary terminals, and the secondary circuit is open, the primary no load current C_0 sets up a magnetic potential difference which accounts for two fluxes—

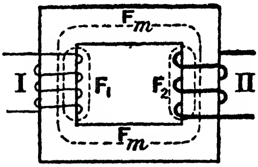

FIG. 48.—Flux circuits of the transformer.

(1) That linking primary and secondary, called the mutual induction F_m.

(2) That linking the primary alone, called the primary leakage flux or self-induction F_1.

The vector sum of these two fluxes (which are often very nearly in phase) sets up in the primary an E.M.F. which exactly balances E minus the pressure absorbed by the primary resistance (C_0r_1). F_m alone sets up the secondary open circuit voltage E_2.

When current is taken from the secondary, another flux makes its appearance, viz. the secondary leakage flux or self-induction F_2 linked with the secondary turns alone, just as F_1 is linked with the primary.

At the same time there appears, of course, in the primary circuit the component of current which counterbalances the secondary ampere-turns, so that C_0 changes to $(C_0 + C_s)$, *i.e.*

to C_1, where C_s is the primary current component, balancing the secondary load current. It is the vector sum of primary and secondary ampere-turns which now produces the flux F_m, and its value is determined from the consideration that, the E.M.F. set up in the primary by F_m, plus that due to F_1, plus the primary resistance drop C_1r_1, must balance the primary applied E.M.F., for these are the only E.M.F.'s existing in the primary circuit.

Since F_1 and C_1r_1 are both proportional to C_1, it follows that so long as E (the applied E.M.F.) is constant, F^m must decrease as C_s increases. Thus when more current is called for on the secondary side, the transformer may be said to automatically "unchoke" itself, so that by reduction of F_m, C_1 may increase.

Admittance and Reactance.—Our view of the relationship between F_1, F_m, F_2, so long as saturation of the iron does not exist, is this :—

(1) That with constant $\sim$, F_1 may be treated as proportional to the primary current alone. Hence the E.M.F. set up in the primary by F_1 is proportional to C_1, and has a real value c_1x_1, where x_1 is constant and is called the primary reactance. Thus (see p. 83)

$$x_1 = 2\pi \sim l_1 = \frac{2\pi \sim t_1F_1}{\sqrt{2}C_1 \times 10^8}$$

where $l_1 = \dfrac{t_1F_1}{\sqrt{2}C_1 \times 10^8}$ is the "coefficient of self-induction" of the primary coil; and "self-induction" is the leakage flux explained above.

(2) That with constant $\sim$, F_2 may be treated as proportional to the secondary current C_2 alone. Thus the E.M.F. set up in the secondary by C_2 has a real value c_2x_2 where x_2 is again constant.

Also $$x_2 = 2\pi \sim l_2 = \frac{2\pi \sim t_2F_2}{10^8 \times C_2\sqrt{2}}$$

where $l_2 = \dfrac{t_2F_2}{\sqrt{2}C_2 \times 10^8}$ is the coefficient of self-induction of the secondary circuit, the self-induction being the secondary leakage flux as just explained.

(3) That with constant ~, F_m may be treated as proportional to the sum of the primary ampere-turns and the secondary ampere-turns; and therefore proportional to the difference $(C_1 - C_s)$, where C_s is that primary current which produces ampere-turns equal and opposite to the secondary ampere-turns. In conformity with (1) and (2) above, and in accordance with the principles expressed in Chap. IV., we might write the E.M.F. produced in the primary by this flux (see p. 97)

$$(C_1 - C_s)Z_m$$

where Z_m is constant and such that $z_m = f_m\sqrt{2}\pi t_1 \sim 10^{-8}$

Voltage Relations.—From these premises we now proceed to express the E.M.F.'s due to these various inductions as vectors.

(1) c_1x_1 is an E.M.F. 90° behind C_1. It may be written therefore jC_1x_1. The component of the applied E.M.F. which balances this or is opposed by it is $-jC_1x_1$.

(2) In like manner we obtain jC_2x_2 and $-jC_2x_2$.

(3) In strict conformity with the above we have $j(C_1 - C_s)Z_m$ as the back E.M.F. in the primary due to the flux F_m; and the E.M.F. applied to the primary which balances the back E.M.F. we call E_1, so that (see equation (73))

$$E_1 = (C_1 - C_s)Z_m = (C_1 - C_s)(r_m - jx_m) \quad . \quad (80)$$

$(C_1 - C_s)$ is then the current required to magnetize the iron of the transformer. It may be called C_0.

So that C_0 has $\frac{e_1}{z_m}$ as its arithmetical value, and

$$C_0 = E_1Y \quad . \quad . \quad . \quad . \quad . \quad . \quad (80a)$$

where Y is the admittance of the primary winding, as determined by the iron circuit on which it is wound (p. 81).

In the equation, then

$$E_1 = C_0 Z_m = (C_1 - C_s) Z_m$$

where $$Z_m = \frac{1}{Y} = (r_m - j x_m)$$

Z_m, r_m, and x_m all accord with the use of these letters in simple circuits (Chap. IV.); r_m, though it has the dimensions of resistance and may be measured in ohms, is not truly physically a resistance.

It is clear that wherever Y occurs in an equation, $\frac{1}{Z_m}$ may be substituted; and often this substitution is most convenient, especially where apparatus is connected in series.

The current C_0 has a component C_i such that the product $E_1 C_i$ gives the watts to be supplied to provide for the iron losses; and C_i is in phase with E_1, consequently 90° ahead of the other component of C_0, viz. C_m. So vectorially we write

$$C_i + C_m = C_0 = E_1 (g + jb) = E_1 Y$$

where $$E_1 g = C_i \text{ and } jE_1 b = C_m$$

C_0 is thus seen to depend on the value of the back E.M.F.

Returning now to x_1 and x_2, we have

$$\text{E.M.F. absorbed by } c_1 x_1 = -j C_1 x_1$$

$$\text{,,} \qquad \text{,,} \qquad c_1 r_1 = C_1 r_1$$

Voltage applied to the primary terminals to drive the current against the impedance of the windings

$$= C_1 Z_1 = C_1 (r_1 - j x_1)$$

Similarly, voltage generated in the secondary to drive the current against the impedance of the windings

$$= C_2 Z_2 = C_2 (r_2 - j x_2)$$

And just as the secondary current is represented by a primary current component, so each component of secondary E.M.F. is represented by a primary component. Hence impedances also in either circuit may be translated into equivalent impedances in the other circuit.

We must here make clear what we mean by the ratio of transformation (called k). Strictly speaking, this is the ratio

$$\frac{\text{applied primary volts}}{\text{terminal secondary volts}}$$

If, however, we consider mutual induction alone, we have

$$-e_1 = \frac{4{\cdot}44 t_1 \sim f_m}{10^8}$$

and
$$e_2 = \frac{4{\cdot}44 t_2 \sim f_m}{10^8}$$

so that
$$-\frac{e_1}{e_2} = \frac{t_1}{t_2}$$

and this, the ratio of turns, is more frequently referred to as the ratio of transformation. We adopt this system and write $\frac{t_1}{t_2} = k$.

Equivalent Impedance.—Any secondary E.M.F. e_2 referred to the primary circuit is expressed as ke_2; and any secondary current c_2, if referred to the primary circuit, becomes $\frac{c_2}{k}$. This is obvious, when we remember that the secondary ampere-turns c_2t_2 are represented in the primary by an equal number of ampere-turns; so that

$$c_s t_1 = -c_2 t_2 \text{ and } c_s = -c_2 \frac{t_2}{t_1} = -\frac{c_2}{k}$$

In transferring an impedance E.M.F. from one circuit to the other we have the following relationship:—

Let a secondary E.M.F. $V_2 = C_2Z_2$ where C_2 is the secondary current.

Then this E.M.F. will in the primary circuit be represented by an E.M.F. V_1, where

$$\begin{aligned} V_1 &= kV_2 \\ &= kC_2Z_2 \end{aligned}$$

Also C_2 will be represented in the primary circuit by a current $-C_s$, such that

$$-C_s = \frac{C_2}{k} \text{ or } C_2 = -kC_s$$

Substituting, we get

$$V_1 = -k^2 C_s Z_2$$

So that an impedance voltage represented in the secondary by the product of a secondary current and the secondary impedance must be represented in the primary circuit by the product of the corresponding primary component of current, and the secondary impedance multiplied by k^2.

$Z_1 + k^2 Z_2$ is called the total "equivalent" impedance of the transformer; and is of importance because, when multiplied by the primary current, it gives the total pressure drop in the transformer. By similar reasoning we obtain $Z_1 \div k^2$ as the equivalent impedance in the secondary circuit of a primary impedance Z_1.

It is evident from this that the transformer may be considered as consisting of one electric circuit alone, provided that we include in this circuit the secondary impedances multiplied by k^2, and remember—

(1) That the whole primary current passes through the primary coils, and is therefore concerned with the primary impedance.

(2) That only that component of the primary current which counterbalances the secondary current is concerned with the equivalent secondary impedance in the primary circuit.

These points will be emphasized in the example which follows.

We may here add the following interesting considerations with respect to this equivalent impedance:—

If the equivalent of Z_2 in the primary circuit is $k^2 Z_2$, it follows that the equivalents of r_2 and x_2 will be $k^2 r_2$ and $k^2 x_2$ respectively.

Now, in any coil of given cross-section and diameter, the number of turns is nearly inversely proportional to the area of the insulated wire. Thus $t \propto \frac{1}{a}$, where a is the area of the insulated wire, and t the number of turns of the winding.

The resistance of the coil is proportional to $\frac{t}{a}$, *i.e.* proportional to t^2. Thus, in considering the effect of resistance as transferred from the secondary circuit to the primary, we shall have to multiply r_2 by $\frac{t_1^2}{t_2^2}$ *i.e.* by k^2.

Again, we know by definition that

$$x_2 \propto t_2^2 \text{ (p. 78)}$$

That is, in considering the effect of reactance as transferred from the secondary circuit to the primary, we shall have to multiply x_2 by $\frac{t_1^2}{t_2^2}$, *i.e.* by k^2.

Thus a second and physical proof is afforded of the relationship between impedance in the primary and secondary circuits (*cf.* the equivalent electrical circuits, p. 227).

General Equations of the Transformer.—In the transformer, as has been said, the volt-amperes appearing in the secondary circuit (whether useful or not) are all supplied from the primary through the medium of the magnetic flux. It is necessary, therefore, to consider the relationship which exists between the equations of the two circuits, and also to carefully distinguish between the secondary voltage and current E_2 and C_2, and the voltage and current which in the primary circuit represent E_2 and C_2. In other words, it is necessary to realize the difference between E_2 and a voltage which, applied to the primary circuit, produces that magnetic flux setting up E_2; and also between C_2 and that current which, flowing through the primary windings, produces ampere-turns to counterbalance those due to C_2 in the secondary coils. We can best approach this relationship by considering the secondary circuit independently, afterwards tracing its influence upon the primary input. We know (1) that the secondary current C_2 sets up ampere-turns producing through the transformer core a magnetic reaction; (2) this reaction is opposed and balanced by a current in the primary C_s, producing an equal number of ampere-turns; (3) the total primary current C_1 is the vector sum of the current balancing the above reaction and of the current producing the magnetic flux corresponding to E_2, the secondary E.M.F.

Fig. 49 expresses the voltage equation for the secondary circuit. In vector notation this is

$$E_2 = C_2Z_2 + C_2Z \quad . \quad . \quad . \quad . \quad . \quad (81)$$

meaning that E_2 may be resolved into two components, one of which is absorbed by the load impedance and the other by the secondary impedance. These two components are C_2Z and C_2Z_2.

Fig. 50 shows the corresponding equation for the primary current. Here C_1, the primary current, is resolved into two components: (1) that part which balances the reaction due to the secondary current C_2—this is marked C_s, and has a value

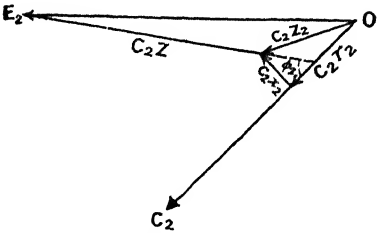

FIG. 49.—Voltage diagram for the secondary circuit.

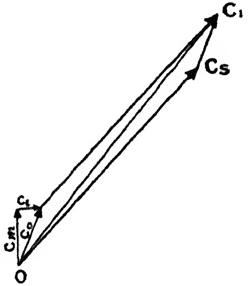

FIG. 50.—Current diagram for the primary circuit.

where k is the ratio of turns $= \frac{\text{primary turns}}{\text{secondary turns}}$; (2) that part which provides the flux producing E_2, marked C_0. Thus

$$C_1 = C_s + C_0 = -\frac{1}{k}C_2 + C_0 \quad . \quad . \quad . \quad (82)$$

Fig. 51 shows the diagram for the equation to the primary E.M.F. In it E, the applied primary pressure, is resolved into two components: (1) that which balances the back E.M.F., called E_1, where $E_1 = -kE_2$; (2) that part which is absorbed by the impedance of the primary windings, written C_1Z_1. In vector notation the equation is

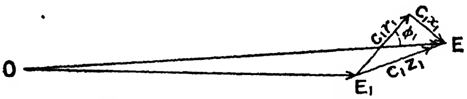

Fig. 51.—Voltage diagram for the primary circuit.

$$E = E_1 + C_1Z_1 = -kE_2 + C_1Z_1 \quad . \quad . \quad (83)$$

Expanding equation (83) by means of (82), we get

$$E = E_1 + C_sZ_1 + C_0Z_1 \quad . \quad . \quad . \quad . \quad (84)$$

Since $E_1 = -kE_2$, and $C_s = -\frac{1}{k}C_2$, we may write equation (81)

$$E_1 = k^2C_sZ_2 + k^2C_sZ$$

Substituting in (84)

$$E = k^2C_sZ_2 + k^2C_sZ + C_sZ_1 + C_0Z_1 \quad . \quad . \quad (85)$$

The meaning of C_s must never be forgotten. It is a vector opposite in direction to C_2 of an arithmetical value $\frac{1}{k}$ times that of C_2.

Equation (85) *is the general equation for every form of alternate-current transformer.* In it the relationship between an impedance in the primary and an impedance in the secondary is clearly expressed. Z_2 is a number of ohms in the secondary circuit, so also is Z. C_s is a current in the primary circuit. In order to express the pressure drop in the primary, due to Z_2, we find it necessary to multiply Z_2 by k^2 as shown.

Returning to equation (85), we may call attention to the fact that C_0Z_1 is usually a small quantity—so small that many neglect it altogether. It is, however, the very advantage of vector algebra that it is quite easy to take everything into account.

We may now select some axis of reference for the vector diagram. In choosing such an axis it will naturally be found advantageous to select that vector most often occurring in the equation to be expressed, for care in this respect often simplifies the resulting expressions. In this instance it is, therefore, an advantage to adopt C_s as the axis of reference. We take it as horizontal, and mark it as a vector c_s'. The actual value of this vector is

$$c_s = \frac{1}{k}c_2 \text{ amperes}$$

In order to find the value of e, we shall need C_0 in terms of either C_2 or C_s. We know that

$$C_0 = E_1Y = k^2C_s(Z_2 + Z)Y \text{ [1]}$$

which means that if we operate upon the vector C_s first

---

[1] Since $\frac{1}{Z_m}$ may always be substituted for **Y** (see p. 81), it is evident that the corresponding voltage equation may be written either as

$$C_0Z_m = E_1 = -k^2E_2 = k^2C_s(Z_2 + Z)$$

or as

$$(C_1 - C_s)Z_m = E_1 = -k^2E_2 = k^2C_s(Z_2 + Z)$$

Whence also we obtain the relationship of primary to secondary current

$$\frac{C_1}{C_s} = \frac{k^2(Z_2 + Z) + Z_m}{Z_m} = \frac{\text{total secondary impedance} + \text{mutual impedance}}{\text{mutual impedance}}$$

$$= 1 + \frac{\text{total secondary impedance}}{\text{mutual impedance}}$$

with the complex number $(Z_2 + Z)$ and afterwards with the complex number Y, we obtain the vector C_0, representing the current supplied to the transformer for magnetizing and for iron losses.

This expression is very interesting as showing the connection between load and magnetizing current. Substituting in (85) this value of C_0, we get

$$E = k^2C_sZ_2 + k^2C_sZ + C_sZ_1 + k^2C_s(Z_2 + Z)YZ_1 \qquad (86)$$

All the vectors which compose E are now in terms of C_s, and we shall proceed to work out an example to show what each term means, and what their relative magnitudes are.

Example to illustrate the Use of the Symbolic Equations for obtaining the Vector Relations in an Actual Transformer.—A core-type single-phase 50-frequency transformer has the following constants: net core section, 20 square inches; length of mean magnetic line in the core, 60 inches; maximum magnetic density, 45,000 lines per square inch; primary turns, 1500 of resistance $r_1 = 7$ ohms; secondary turns, 100 of resistance $r_2 = 0{\cdot}025$ ohm; primary reactance, 14 ohms; secondary reactance, 0·036 ohm. Find the symbolic expression for the no-load admittance, and assuming the flux common to primary and secondary as constant, give the equations for, and value of, the primary input corresponding to a secondary output of 20 amperes (non-inductive load).

From the above dimensions we have: core weight = 324 lbs. approximately. With 50 frequency and the density given, the iron loss will be about one watt per pound. Hence iron loss = 324 watts.

45,000 lines per square inch needs six ampere-turns per inch in good iron (imbricated joints).

Hence primary magnetizing ampere-turns maximum $= 6 \times 60 = 360$.

Primary magnetizing idle current

$$= \tfrac{360}{1500} = 0{\cdot}24 \text{ ampere (maximum)}$$
$$= 0{\cdot}17 \text{ ampere R.M.S.}$$

Now the maximum flux $= 45{,}000 \times 20 = 9 \times 10^5$ lines.

Hence primary volts (induced)

$$= \frac{4{\cdot}44 \times 1500 \times 50 \times 9 \times 10^5}{10^8}$$
$$= 3000 = e_1$$

Energy current for iron loss $= \frac{324}{3000} = 0{\cdot}108$ ampere. From definition of no-load conductance and susceptance—

$$\text{Energy current} = ge_1 = 0{\cdot}108 \text{ ampere}$$
$$\therefore g = \frac{0{\cdot}108}{3000} = 0{\cdot}000036 \text{ mho.}$$
$$\text{Idle current} = be_1 = 0{\cdot}17 \text{ ampere}$$
$$\therefore b = \frac{0{\cdot}17}{3000} = 0{\cdot}000057 \text{ mho.}$$

Whence $Y = g + jb = 0{\cdot}000036 + j0{\cdot}000057$
$=$ primary admittance on no load

If the flux common to the two coils is constant, so also is the magnetizing idle current and the iron loss energy current.

Ratio of turns $= 15 = k$
Secondary induced volts $e_2 = \frac{3000}{15} = 200$

The current $c_s = \frac{1}{k}c_2 = \frac{1}{15} \times 20 = \frac{4}{3}$ ampere

Voltage Calculation.—We shall now proceed to make use of equation (86), and for convenience we shall consider the parts of this expression independently. Writing out the expressions for the impedances in full, we have

$$Z_1 = r_1 - jx_1 = 7 - j14$$
$$Z_2 = r_2 - jx_2 = 0{\cdot}025 - j0{\cdot}036$$

Also $Z_2 + Z = (r_2 + r) - j(x_2 + x) = 0{\cdot}025 + r - j(0{\cdot}036)$

for the load is non-inductive—*i.e.* $x = 0$.

From equation (81) we get with C_2 as axis of reference

$$E_2 = c_2'(Z_2 + Z)$$

Substituting the values already obtained

$$E_2 = c_2'(0{\cdot}025 + r) - 0{\cdot}036c_2''$$

The real value of c_2 being 20 amperes and of e_2 200 volts, we get

$$200 = (0{\cdot}5 + 20r)' - (0{\cdot}72)''$$

from which by the usual process we get the real value of r.

$$\sqrt{(0{\cdot}5 + 20r)^2 + (0{\cdot}72)^2} = 200$$
$$r^2 + \tfrac{1}{20}r = 100 - 0{\cdot}00192$$

The last figure is obviously negligible in this case. This shows that the secondary resistance is very small compared with the load resistance r, which should always be the case in well-designed transformers, so that

$$r = 9{\cdot}999 \text{ ohms, or practically } 10 \text{ ohms}$$

Also, if ϕ_2 = the angle of lag of the secondary current behind the secondary induced E.M.F.

$$\tan\phi_2 = \frac{x_2}{r_2 + r} = \frac{0{\cdot}036}{10{\cdot}025} = 0{\cdot}00359, \text{ and } \phi_2 = \text{about } 0{\cdot}2^\circ$$

It is hardly necessary to call attention to the impossibility of measuring such an angle from a diagram.

$$\text{Terminal volts} = c_2z = c_2r = 199{\cdot}9$$

We have, then, from the first part of the expression for E in equation (86)

$$\begin{aligned}
&k^2C_s.(Z_2 + Z) + C_sZ_1\\
&\quad = k^2C_s\{(10{\cdot}025) - j0{\cdot}036\} + C_s(7 - j14)\\
&\quad = C_s(2263 - j22{\cdot}1)\\
&\quad = c_s'.2263 - c_s''.22{\cdot}1\\
&\quad = (\tfrac{4}{3}.2263)' - (\tfrac{4}{3}.22{\cdot}1)''\\
&\quad = (3016)' - (29{\cdot}5)''
\end{aligned}$$

The second part of E in equation (86) becomes

$$k^2C_s(Z_2 + Z)YZ_1 = C_s(2256 - j8{\cdot}1)YZ_1$$
$$= C_s\,.\,(2256 - j8{\cdot}1)\left(\frac{36}{10^6} + j\frac{57}{10^6}\right)(7 - j14)$$

This may be multiplied out according to the ordinary rules for multiplication of complex quantities (see p. 49), and gives

$$C_s(2{\cdot}368 - j0{\cdot}245) = (3{\cdot}114)' - (0{\cdot}327)''$$

This is the voltage necessary to maintain the magnetizing current through the impedance of the primary winding. It is obvious that this might be obtained from the expression C_0Z_1, where C_0 has the value already worked out on p. 112.

The complete expression for E is then

$$\begin{aligned} E &= (3016)' - (29{\cdot}5)'' + (3{\cdot}114)' - (0{\cdot}327)'' \\ &= (3019)' - (29{\cdot}83)'' \end{aligned}$$

So that $\quad e = \sqrt{(3019)^2 + (29{\cdot}83)^2} = 3020$ volts

So for a load of 20 amperes on the secondary of this transformer the applied voltage must be raised from 3000 to 3020 if the flux is to be kept constant. At this load the applied voltage leads the load current C_s by an angle such that

$$\tan \phi_1 = \frac{29{\cdot}83}{3019} = 0{\cdot}01 \text{ (nearly)}$$
$$\phi_1 = 30 \text{ minutes (about)}$$

It will be noticed that, by neglecting the second part of equation (86), the primary voltage would come out 3018, or about 0·06 per cent. low. This is, of course, less than the error of a very good voltmeter, so that neglect of this factor is usually justifiable, and the transformer equations are very considerably simplified; for equation (86) becomes

$$E = k^2C_s(Z_2 + Z) + C_sZ_1 \quad . \quad . \quad . \quad (86a)$$

which is very easy to work out, and quite near enough for most practical purposes.

Current Calculation.—The total primary current in the transformer is

$$C_0 + C_s$$

C_s we already have as $\frac{4}{3}$ amperes; C_0 we also have in the form

$$C_0 = E_1(g + jb) = 0{\cdot}108' + 0{\cdot}17''$$

That is, we have C_0 in terms of E_1 as axis of reference. Now, in order to add directly C_0 and C_s we must have both C_0 and C_s referred to one axis. We chose C_s as the axis of reference and wrote it $= \frac{4}{3}'$. As C_0 in the form given above is in terms of E_1 as axis of reference, it cannot be added directly to C_s.

But it has already been shown that

$$\begin{aligned} C_0 &= k^2 C_s(Z_2 + Z)Y \\ &= C_s(2256 - j8{\cdot}1)\left(\frac{36}{10^6} + j\frac{57}{10^6}\right) \\ &= C_s(0{\cdot}08 + j0{\cdot}128) \\ &= (\tfrac{4}{3}\,0{\cdot}08)' + (\tfrac{4}{3}\,0{\cdot}128)'' \\ &= 0{\cdot}107' + 0{\cdot}171'' \end{aligned}$$

which is very near the value given above, showing that E_1 and C_s are nearly in phase. We get now

$$\begin{aligned} C_1 &= C_s + C_0 \\ &= \tfrac{4}{3}' + 0{\cdot}107' + 0{\cdot}171'' \\ &= 1{\cdot}44' + 0{\cdot}171'' \end{aligned}$$

Hence $$c_1 = \sqrt{(1{\cdot}44)^2 + (0{\cdot}171)^2} = 1{\cdot}45 \text{ amperes}$$

The tangent of the angle of lag of C_1 behind C_s

$$= \tan\phi_3 = \frac{0{\cdot}171}{1{\cdot}44} = 0{\cdot}119$$

$$\phi_3 = 6^\circ \text{ (nearly)}$$

And it has already been shown that the applied voltage leads C_s by 30 minutes, or $\frac{1}{2}°$. Hence the applied voltage leads the total primary current C_1 by

$$\phi = \tfrac{1}{2}° + 6° = 6\tfrac{1}{2}°$$

So that the power factor $= \cos\phi = 0{\cdot}99$.

EXAMPLE.—*Transformer worked at constant applied Pressure.*—The case just considered is a very simple one, which on the one hand does not often occur in practice, and on the other does not exemplify the use of the change of axis of reference or the case of inductive secondary load. We will therefore work out the behaviour of this same transformer under the following conditions:—

Give the primary current and the secondary pressure at no load and at a load of 30 amps. with a power factor of 0·5 and a constant applied primary pressure of 3000 volts.

No Load.—Now at no load $C_2 = C_s = 0$ and Z becomes infinite. Equation (84) therefore is

$$E = E_1 + C_0 Z_1$$

Now $$C_0 = E_1 Y$$

so $$E = E_1(1 + YZ_1)$$

adopting E_1 as the axis of reference

$$E_1 = e_1'$$

then $$E = e_1' + e_1'YZ_1$$

Now we have already pointed out how small $C_0 Z_1$ usually is. In the present instance, if we neglect it, we get

$$E = e_1'$$

If we took it into consideration we should get

$$E = e_1'(1{\cdot}001) - e_1''(0{\cdot}0001)$$

from which it is evident that to consider the e_1'' term would be to carry accuracy to the verge of absurdity.

E then has an arithmetical value

$$e = e_1 = 3000$$

and

$$e_2 = \frac{1}{k}e_1 = 200$$

No-load Current.—As in the previous example with E_1 as axis of reference this is (from p. 115)

$$\begin{aligned} C_0 &= E_1Y = e_1'g + e_1''b \\ &= (0{\cdot}108)' + (0{\cdot}17)'' \end{aligned}$$

Its real value is

$$c_0 = \sqrt{(0{\cdot}108)^2 + (0{\cdot}17)^2} = 0{\cdot}2 \text{ amps.}$$

Note that c_0 has its greatest value at no load, because then e_1 is greatest. If C_0Z_1 is negligible at no load, it is almost always so on load. For load conditions, then, equation (86) is with sufficient accuracy (86*a*), *i.e.*

$$E = k^2C_s(Z_2 + Z) + C_sZ_1$$

Load Conditions.—In considering the load conditions we must first ascertain the value of Z. This may be done by substitution in equation (86*a*).

We have $c_s = \frac{1}{k}c_2 = \frac{1}{15}30 = 2$ amps.

$k^2 = 225 \qquad Z_2 = 0{\cdot}025 - j0{\cdot}036$

Z unknown. $Z_1 = 7 - j14$

$$E = k^2C_sZ_2 + k^2C_sZ + C_sZ_1$$

Taking C_s as axis of reference

$$\begin{aligned} E &= (11{\cdot}25)' - (16{\cdot}2)'' + 450'Z + (14)' - (28)'' \\ &= (25{\cdot}25)' - (44{\cdot}2)'' + (450)'Z \end{aligned}$$

Value of Z.—Since for the load circuit

$$\cos\phi = 0{\cdot}5 = \frac{r}{\sqrt{r^2+x^2}}$$

we have $$0{\cdot}25(r^2+x^2) = r^2$$

whence $$x = 1{\cdot}73r$$

and $$Z = r - jx = r - j1{\cdot}73r$$

Substituting this in the equation for E, we get

$$\mathbf{E} = (25{\cdot}25)' - (44{\cdot}2)'' + (450r)' - (780r)''$$

Putting in the value for E (= 3000 volts) and rearranging the terms

$$(25{\cdot}25 + 450r)' - (44{\cdot}2 + 780r)'' = 3000$$

Squaring the parallel and the normal components respectively, and adding them, we get a quadratic in r

$$r^2 + 0{\cdot}113r = 11{\cdot}1$$

Since r is of necessity positive, we have the solution

$$r = 3{\cdot}27 \text{ ohms}$$

and $$Z = r - jx = r - j1{\cdot}73r$$
$$= 3{\cdot}27 - j5{\cdot}66$$

and the real value of z is

$$\sqrt{(3{\cdot}27)^2 + (5{\cdot}66)^2} = 6{\cdot}54 \text{ ohms}$$

Load P.D. and Voltage Drop.—The former is the product of load impedance and current. Thus

$$\text{Load P.D.} = 30 \times 6{\cdot}54 = 196{\cdot}2 \text{ volts}$$

So between no load and 30 amperes (with $\cos\phi = 0{\cdot}5$) the voltage varies between 200 and 196·2. The student may with advantage find the error introduced by neglecting the last term of equation (86) in this calculation.

Total Impedance in the Secondary Circuit.—This is the sum of Z_2 and Z.

$$
\begin{aligned}
&= r_2 - jx_2 + r - jx \\
&= 0{\cdot}025 - j0{\cdot}036 + 3{\cdot}27 - j5{\cdot}66 \\
&= 3{\cdot}295 - j5{\cdot}696
\end{aligned}
$$

Secondary E.M.F.—The total secondary E.M.F. is the product of secondary current and total secondary impedance. Taking C_2 as axis of reference

$$
\begin{aligned}
E_2 &= C_2(Z_2 + Z) \\
&= 30' \times (3{\cdot}295 - j5{\cdot}696) \\
&= 98{\cdot}85' - 170{\cdot}9'' \\
e_2 &= \sqrt{(98{\cdot}85)^2 + (170{\cdot}9)^2} = 197{\cdot}3 \text{ volts} \\
e_1 &= 197{\cdot}3 \times k = 197{\cdot}3 \times 15 = 2960 \text{ volts}
\end{aligned}
$$

Value of C_0 *on Load.*

$$C_0 = E_1 Y$$

The value of Y is $0{\cdot}000036 + j0{\cdot}000057$ (see p. 112).

$$
\begin{aligned}
C_0 &= 2960'(0{\cdot}000036 + j0{\cdot}000057) \\
&= 0{\cdot}1066' + 0{\cdot}1686'' \\
c_0 &= 0{\cdot}1994 \text{ amps.}
\end{aligned}
$$

It will be remembered that the value of C_0 at no load was

$$0{\cdot}107' + 0{\cdot}17''$$

of real value **0·2** amps.

So the above figures show the small reduction of the current C_0 due to load.

Value of Total Primary Current.

$$
\begin{aligned}
C_1 &= C_s + C_0 \\
&= C_s + k^2 C_s (Z_2 + Z) Y
\end{aligned}
$$

Taking C_s as axis of reference, and remembering that

$$e_1 = -ke_2$$

we have

$$\begin{aligned} C_1 &= C_s' + k(98{\cdot}85' - 170{\cdot}9'')(0{\cdot}000036 + j0{\cdot}000057) \\ &= 2' + 0{\cdot}2' - 0{\cdot}008'' \\ c_1 &= \sqrt{(2{\cdot}2)^2 + (0{\cdot}008)^2} = 2{\cdot}2 \text{ amps.} \end{aligned}$$

It is evident from this value that C_s and C_1 are practically in phase.

Power Factor.—From equation ($86a$)

$$E = k^2C_s(Z_2 + Z) + C_sZ_1$$

With C_s as axis of reference this is

$$\begin{aligned} E &= 1490' - 2550'' + 2'(7 - j14) \\ &= 1504' - 2578'' \end{aligned}$$

That is, the component of terminal E.M.F. in phase with C_s is 1504 volts; that at right angles to C_s is 2578 volts.

So that $\cos\phi = \frac{1504}{3000} = \mathbf{0{\cdot}501}$ (approximately)

We here neglect the small angle between C_s and C_1.

CHAPTER VI

MOTORS OF THE INDUCTION TYPE

As already stated, every alternating-current apparatus depending upon magnetic induction for the transference of electrical energy, involves the principle and action of the transformer. With certain modifications to suit each individual case, the equations deduced in the preceding chapter apply directly to polyphase and single-phase induction motors, and motors of the repulsion and repulsion-induction type.

The object of the following pages is to show how the transformer equations are changed for certain typical cases.

Generally speaking, there are only two types of induction motor, viz. (1) that in which the coils are closed upon themselves without the intervention of a commutator; and (2) that in which the coils upon the rotor are connected to a commutator with short-circuited collecting brushes. Either of these types may be single-phase or polyphase. To the first type belong the monophase and polyphase induction motors, and to the second the monophase repulsion motors.

In either case we have the same general conditions existing, viz. one or more alternating fields set up by alternating currents of one or more phases in the stator, acting upon one or more closed circuits in the rotor. The simplest form of the machine is evidently a bipolar laminated magnet excited by a simple alternating current, between the poles of which there is a rotor bearing a series of single closed coils.

Induced Voltages in the Rotor.—If we consider the general case of a squirrel-cage rotor moving in such a single-phase sinusoidally distributed field I, I (Fig. 52), it will be seen that the rotor conductors, when revolving, may be considered in two ways.

Firstly, they may be considered as being constantly joined up by the end rings in pairs on either side of an axis AB, so that they form vertical short-circuited secondary coils to a transformer of which the poles with their coils I, I form the primary. In which case the average R.M.S. value of the E.M.F. induced in any rotor turn is

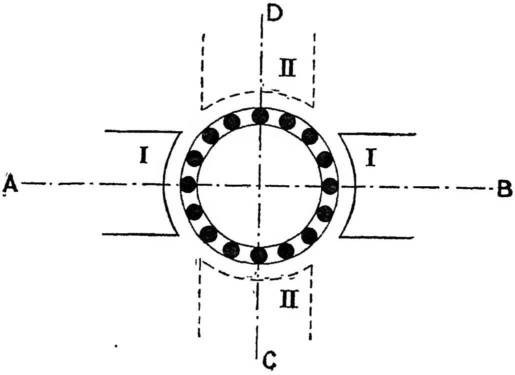

FIG. 52.—Rotor and field of induction motor.

$$e_2=\sqrt{2}\pi \times \text{maximum value of flux} \times \sim \times \text{constant} \times 10^{-8}$$

where the constant takes into consideration the fact that the turns are not concentrated into a single coil, but distributed around the rotor periphery. No E.M.F. due to rotation can be developed in coils having the axis AB.

Secondly, the rotor bars may be imagined as being constantly paired on either side of an axis CD; in which position, though they cannot form closed secondary coils to the primary I, I, they yet are in the same position with regard to the field poles I, I, as are the armature conductors of a direct-current dynamo; and consequently when rotating they generate an E.M.F. of R.M.S. value

$$v_2 = \begin{cases} \sqrt{2}\pi \times \text{maximum value of flux} \times \text{revs. per} \\ \qquad \text{sec.} \times \text{constant} \times 10^{-8} \end{cases}$$

Now the constant has the same value in the two cases; for it but allows for the distribution of the conductors about the rotor periphery. Thus

$$e_2 : v_2 = \sim : n$$

where n = frequency of rotation in cycles per second.

Thus these two E.M.F.'s may be written E_2 and $E_2\frac{n}{\sim}$.

As regards phase, E_2 is at right angles in time to the flux, but exists in coils about the same axis with it in space. $E_2\frac{n}{\sim}$ is in phase with the flux, but exists in coils about an axis at right angles to it in space.

These two electromotive forces are to be found in all forms of monophase motors. An E.M.F., however, such as $E_2\frac{n}{\sim}$ in coils along an axis at right angles to that of the stator coils could of itself not have any connection with the static transformer primary I, I, were it not that this E.M.F. can set up a magnetizing current producing a flux at right angles to AB in space, and lagging 90° behind $E_2\frac{n}{\sim}$ in time.

In this flux the coils about the axis AB must be considered as rotating; and they will consequently have in them an E.M.F. induced in phase with this flux, and of value $E_2\frac{n}{\sim} \times \frac{n}{\sim}$. This, of course, will have a counterbalancing component in the primary circuit, just as E_1 counterbalances E_2 in the transformer.

The total E.M.F. in the secondary is therefore in this case

$$E_2\left(1 - \frac{n^2}{\sim^2}\right)$$

the negative sign being used because the machine is a motor, and therefore the E.M.F. of rotation must tend to stop the motion.

Polyphase Motors.—The case of the polyphase motor may be considered in rather a different way. Consider the two-phase motor made by adding a second phase to the single-phase machine, and we get the poles represented by a dotted line in Fig. 52.

Here, since phase II is exactly similar but at right angles in time and space to phase I, we can consider the machine as consisting of two transformers, and take all E.M.F.'s acting in coils around the AB axis as having counterbalancing components in phase I on account of their common flux; and coils about the CD axis as having counterbalancing components in phase II for a similar reason.

The flux in phase I sets up, as before, E_2 and $E_2\frac{n}{\sim}$, but the latter now is counterbalanced by an E.M.F. in phase II, because it is set up in coils about CD. Phase II will also set up E_2 and in coils about AB, $E_2\frac{n}{\sim}$. Thus in either case the E.M.F. in the rotor coils corresponding to one phase is

$$E_2\left(1 - \frac{n}{\sim}\right)$$

$$= E_2\left(\frac{\sim - n}{\sim}\right)$$

Now, $\sim - n$ is called the "slip" of the rotor, and we designate it by "*s*."

Thus rotor E.M.F. corresponding to each stator phase

$$= E_2\frac{s}{\sim}$$

$\frac{s}{\sim}$ is called the *fractional slip*.

This reasoning applies to the polyphase motor, whatever the number of phases.

Although the secondary E.M.F. is thus reduced from that of the transformer in the ratio $\frac{s}{\sim}$, the frequency of this

E.M.F. considered with respect to the stator axes evidently remains the same as before; so that where the rotor consists of a D.C. armature and commutator with as many pairs of short-circuited brushes as stator phases, the transformer equation may be applied almost direct at this stage with the substitution merely of $E_2\frac{s}{\sim}$ for E_2 in the secondary.

If, however, the rotor coils are grouped and connected to slip-rings, and these latter short-circuited, the frequency as between slip-rings is not that of the primary.

This may be explained as follows:—

The E.M.F. induced in any turn of a coil is proportional to the flux through which the sides of the coil move and the rate at which they move through it. In an alternating-current system the rate of movement through the flux is proportional to the frequency of the E.M.F. produced in the coil. Now in the two-phase case it has been shown that the resultant E.M.F. of coils acting along either axis is reduced as the speed rises from E_2 to $E_2\frac{s}{\sim}$, yet the flux through which these coils move has been assumed as an alternating flux of constant magnitude along a particular axis. As a result, then, the effect of the rotation of the coil must be to change the rate at which the sides of the coil cut the flux in the ratio of $1 : \frac{s}{\sim}$, *i.e.* the frequency of the E.M.F. in the coil is no longer $\sim$ but $\frac{s}{\sim}(\sim)$. That is, the frequency is equal to the slip, and the secondary impedance becomes $r_2 - j\frac{s}{\sim}x_2$ instead of $r_2 - jx_2$ where x_2 is the reactance of the rotor when stationary.

This effect may be imagined as taking place in either of two ways:—

(1) By considering that the effect of the alternating flux increasing through the coil is discounted by the moving coil presenting less area to the flux.

(2) By considering the resultant stator flux as rotating in

the same direction as the coil, so that the faster the latter moves the less often does it pass through the flux.

Both ideas are quite correct. The latter is particularly useful in explaining many induction-motor problems, and may be proved quite independently of the foregoing.

Rotating Field.—If the alternating fluxes due to the phases of a two-phase machine be denoted by F sin θ and F cos θ respectively, then along any axis spaced from the axis of F cos θ by $\phi°$, the resultant of the two fluxes is

$$\text{F} \sin\theta \sin\phi + \text{F} \cos\theta \cos\phi$$

in the special case in which $\phi = \theta$ we get the resultant

$$= \text{F}(\sin^2\theta + \cos^2\theta) = \text{F}$$

i.e. the resultant field due to the two phases, along an axis making an angle with the axis of one phase equal to the time angle through which the flux of that phase has passed (*i.e.* $\phi = \theta$) is a vector of constant magnitude, and equal to the maximum flux of one phase. In other words, the resultant flux of a two-phase motor is a synchronously rotating flux of constant magnitude equal to the flux of one phase. This may also be shown graphically from the polar diagram of the fluxes.

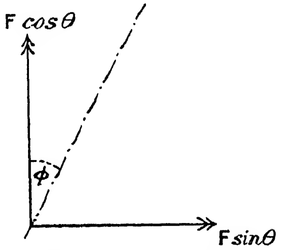

FIG. 53.—Resolution of two-phase flux.

In the same way for a three-phase motor we have, if $\theta = \phi$

$$\text{resultant flux} = \text{F}\{\cos^2\theta + \cos^2(\theta + 120) + \cos^2(\theta + 240)\}$$
$$= 1{\cdot}5\text{F}$$

so that the resultant rotating field in this case is $1\frac{1}{2}$ times the maximum of one phase: and generally the resultant field

$= \frac{q}{2}$ times the maximum of one phase, where q = the number of phases.

From either point of view, then, the polyphase induction motor is seen to have currents of a frequency $\sim$ in the stator and s in the rotor. Thus it is sometimes called a *frequency transformer*.

Analytical Treatment.—The preceding facts are more briefly put in the following analysis :—

Let us imagine that the rotor of a two-phase motor is equipped with a single coil made up of two conductors. If the flux emanating from phase I be F $\sin\theta$, then the part of this flux that is included by the coil depends upon the angle λ, and may be written

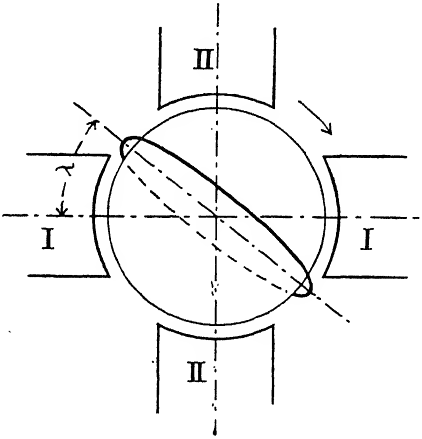

FIG. 54.—Ideal two-phase motor.

$$F \sin\theta \sin\lambda$$

The E.M.F. induced in the coil by this flux is

$$\frac{d}{dt}F \sin\theta \sin\lambda$$

$$= F\left(\cos\theta \sin\lambda\frac{d\theta}{dt} + \sin\theta \cos\lambda\frac{d\lambda}{dt}\right) \quad . \quad . \quad (87)$$

Now $$\frac{d\theta}{dt} = 2\pi \sim$$

and in the two-pole case $$\frac{d\lambda}{dt} = 2\pi \times \text{revs. per sec.}$$
$$= 2\pi n$$

Hence the E.M.F. induced in the turn by this phase is

$$2\pi \sim \mathrm{F}(\cos\theta \sin\lambda) + 2\pi n\mathrm{F}(\sin\theta \cos\lambda) \quad . \quad . \quad (88)$$

This equation we shall refer to later in connection with the single-phase induction motor (see p. 244).

In like manner the flux included by the coil from phase II is

$$\mathrm{F} \cos\theta \cos\lambda$$

for both phase of flux and position of pole are changed by 90°.

The E.M.F. due to this is

$$-2\pi \sim \mathrm{F}(\sin\theta \cos\lambda) - 2\pi n\mathrm{F}(\cos\theta \sin\lambda) \quad . \quad (89)$$

Adding (88) and (89) the resultant E.M.F. in the coil is

$$2\pi\mathrm{F}\{\cos\theta \sin\lambda(\sim - n) - \sin\theta \cos\lambda(\sim - n)\}$$
$$= -2\pi\mathrm{F}(\sim - n)\sin(\theta - \lambda) \quad . \quad . \quad . \quad (90)$$

That is, it is exactly the E.M.F. which would be produced in the secondary coil of an ordinary transformer or in the armature of an ordinary alternator where the frequency was $(\sim - n)$ instead of $\sim$.

Now n represents the revolutions per second of the rotor in the above case, but this is because there are only two poles per phase. With more than two poles per phase n becomes the *frequency due to rotation, i.e.* pairs of poles per phase multiplied by revolutions per second.

It may be objected that equation (87) only refers to a coil in one particular position at a particular time. But this does not affect the general deduction made from equation

(90); for any other coil will simply have a different value of λ, *i.e.* equation (90) will read

$$-2\pi F(\sim - n)\sin(\theta - \lambda + a)$$

where a is the angle between the new and the old coil positions. Thus the general deductions are not changed in the least.

Rotor Impedance.—Returning now to the E.M.F. equations, we see that as the secondary E.M.F. is

$$E_2\frac{\sim - n}{\sim}$$

We can look upon the component $E_2\frac{n}{\sim}$ as almost taking the place of the load C.R. component in the transformer, and instead of transformer equation (81)

$$E_2 = C_2Z + C_2Z_2$$

we might write

$$E_2 = E_2\frac{n}{\sim} + C_2Z_3 \quad . \quad . \quad . \quad . \quad (91)$$

but we must remember that since the frequency of the secondary currents is equal to the slip, s, Z_3 is not constant as Z_2 was, but

$$Z_3 = r_2 - j\,.\,\frac{s}{\sim}x_2$$

instead of $r_2 - jx_2$, as explained on p. 125.

Again, it has been shown that the E.M.F. acting in the rotor is

$$E_2\frac{\sim - n}{\sim}$$

and it is evident that this is all absorbed by the rotor impedance if there is no external resistance.

Hence

$$E_2 \frac{\sim - n}{\sim} = C_2 Z_3$$

which is also expressed in equation (91) above.

Or we may look at the matter of this reduced rotor E.M.F. in still another way.

Since the rotor E.M.F. $= E_2 \frac{\sim - n}{\sim}$

the rotor current per phase will be

$$E_2 \frac{\sim - n}{\sim} \div Z_3$$

$$= E_2 \frac{\sim - n}{\sim} \div \left(r_2 - j\frac{s}{\sim} x_2\right)$$

$$= E_2 \left(\frac{\sim - n}{\sim} \times \frac{1}{r_2 - j\frac{\sim - n}{\sim} x_2}\right)$$

$$= E_2 \div \left(\frac{\sim}{\sim - n} r_2 - jx_2\right) \quad . \quad . \quad . \quad . \quad . \quad (91a)$$

i.e. *the motor acts precisely as a transformer whose secondary resistance increases as the speed rises; or in other words, the effect of speed may be considered simply as changing the secondary impedance from*

$$r_2 - jx_2$$

to

$$\frac{\sim}{\sim - n} r_2 - jx_2$$

Whichever of these alternative views we adopt we get the same result so far as transformer equation (87) is concerned, though the form is slightly different; for we may either write

$$E_2 = E_2 \frac{n}{\sim} + C_2 Z_3 \quad . \quad . \quad \text{(equation (91))}$$

or

$$E_2 = C_2 Z_4 \quad . \quad . \quad . \quad . \quad . \quad \text{(equation (91}a\text{))}$$

where

$$Z_4 = \left(\frac{\sim}{\sim - n} r_2 - jx_2\right)$$

Transformer equations (82), (83), and (84) (p. 109) remain as before.

Equation (85) then becomes either from (91)

$$E = k^2 C_s Z_3 - k E_2 \frac{n}{\sim} + C_s Z_1 + C_0 Z_1$$

or from (91*a*) $E = k^2 C_s \left(\frac{\sim}{\sim - n} r_2 - j x_2 \right) + C_s Z_1 + C_0 Z_1$ (92)

Equation (85), however, was derived specially for use in transformer problems, where E, C, and k are usually known.

In motor problems, other forms of the equation are often more useful. Thus we may know sufficient about the machine to estimate the magnetizing idle and energy currents and the output, and may wish to find the input, power factor, etc.

Now it is noticeable with reference to the expression

$$\begin{aligned} C_0 Z_1 &= E_1 (g + jb)(r_1 - j x_1) \\ &= k^2 C_s \left\{ \left(\frac{\sim}{\sim - n} \right) r_2 - j x_2 \right\} (g + jb)(r_1 - j x_1) \end{aligned}$$

that in average motors C_0 = about $\frac{1}{3}$ of C_s at full load; and $C_s Z_1$ averages about 10 per cent. of the total terminal volts per phase. Hence $C_0 Z_1$ averages from 2 per cent. to 3 per cent. of the volts per phase. While this is much greater than the corresponding value for transformers and thus must not be neglected entirely, it is still rarely worth while estimating it to such great accuracy.

The following is the most useful approximation :—

Assume in the expression for C_0

$$k^2 C_s \left(\frac{\sim}{\sim - n} r_2 - j x_2 \right) = E$$

instead of E_1, then C_0 is known and becomes a constant.

Thus $$C_0 = E(g + jb)$$

which is usually quite near enough, and $C_0 Z_1$ becomes

$$E(g + jb)(r_1 - j x_1)$$

a comparatively simple expression.

Accepting this modification, (92) will read

$$\mathrm{E}\{1-(g+jb)(r_1-jx_1)\}$$
$$=k^2\mathrm{C}_s\left(\frac{\sim}{\sim-n}r_2-jx_2\right)+\mathrm{C}_s\mathrm{Z}_1$$
$$=\mathrm{C}_s\left(\frac{k^2\sim}{\sim-n}r_2+r_1-jk^2x_2-jx_1\right)\quad.\quad(93)$$

The left-hand side of this equation is a vector very nearly in phase with E and quite easily calculated.

Multiplying both sides of the equation by C_s the right-hand side may be analyzed as follows:—

$c_s^2k^2\left(\frac{\sim}{\sim-n}\right)r_2$ = total watts delivered to rotor per phase

$c_s^2k^2\left(\frac{n}{\sim-n}\right)r_2$ = output of motor per phase, including friction losses, etc. . . . . (93*a*)

$c_s^2r_1$ = stator watt loss due to load current

and

$$\frac{c_s^2k^2r_2}{\text{output}+c_s^2k^2r_2}=\frac{\sim-n}{\sim}=\text{fractional slip}^{1}$$

EXAMPLE I.—A three-phase induction motor of 6 B.H.P. is supplied with energy at 250 volts, 40 ~. The iron loss of the machine is 240 watts, the magnetizing current at no load

---

[1] Another useful form of the induction-motor equation directly following from the arguments used above and from Chapter IV. is

$$\mathrm{E}=\mathrm{C}_1\mathrm{Z}_1+\mathrm{C}_sk^2\mathrm{Z}_4\quad.\;.\;.\;.\;.\;.\;.\;.\;.\;.\quad(92)$$

where

$$\mathrm{Z}_4=\frac{\sim}{\sim-n}r_2-jx_2$$
$$\mathrm{E}=\mathrm{C}_s\frac{\mathrm{Z}_1(\mathrm{Z}_m+k^2\mathrm{Z}_4)}{\mathrm{Z}_m}+\mathrm{C}_sk^2\mathrm{Z}_4$$
$$=\mathrm{C}_s\mathrm{Z}_1+\mathrm{C}_sk^2\mathrm{Z}_4\left(1+\frac{\mathrm{Z}_1}{\mathrm{Z}_m}\right)$$
$$=\mathrm{C}_s\mathrm{Z}_1+\mathrm{C}_sk^2\mathrm{Z}_4\lambda_1\text{ nearly}$$

This appears more simple than that used above; but it involves a clear understanding of λ_1 and λ_2, and so is not adopted here.

is 2·5 amperes per phase, and the stator winding is delta-connected. Other particulars are

Stator resistance . . .	1·3 ohms per phase
Rotor equivalent resistance .	1·2 ,, ,,
Dispersion coefficient (σ) .	0·1

Find the stator current, power factor, and output of this machine when the slip is 5 per cent.

The magnetizing current is

$$E(g + jb)$$

and of this $eb = 2{\cdot}5$ amperes

$b = \frac{2{\cdot}5}{250} = 0{\cdot}01$

also $eg = \frac{240}{3} \times \frac{1}{250}$

so that $eg = \frac{80}{250} = 0{\cdot}32$ amperes

Dispersion Coefficient.—Of late years the design and testing of induction motors has all been done with reference to a factor σ, called the dispersion coefficient (p. 94). The origin of the use of this figure is to be found in the fact that it controls almost entirely the well-known circle diagram for the motor, and hence is exceedingly convenient when graphical methods are to be adopted.

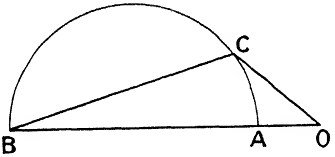

FIG. 55.—Simple circle diagram of induction motor.

In the simple circle diagram (Fig. 55), if OA represent the no-load current of the motor and OB that short-circuit current which would flow if the machine-windings had no resistance and the iron was never saturated, then

$$\sigma = \frac{OA}{OB}$$

Some writers, however, prefer the ratio

$$\frac{OA}{AB}$$

and call this "σ" (*cf.* p. 95).

To avoid confusion we shall adopt the symbols

$$\nu = \frac{OA}{AB}$$

$$\sigma = \frac{OA}{OB}$$

Obviously when one of these coefficients is known the other can be deduced, for

$$\nu = \frac{\sigma}{1 - \sigma}$$

and

$$\sigma = \frac{\nu}{1 + \nu}$$

If OA and AB are drawn to form a straight line, then the iron losses must be supposed either to be neglected or subtracted (see p. 232), so that the current represented by

$$OA = eb$$

Similarly, if the motor has no resistance and the iron is supposed not to be saturated, then on short circuit there is nothing limiting the current but reactance, and that reactance is constant at all loads.

Hence $OB = \dfrac{\text{applied volts}}{\text{equivalent reactance of motor}} = \dfrac{e}{x_1 + k^2 x_2}$[1]

Thus $$\sigma = \frac{OA}{OB} = \frac{eb}{\dfrac{e}{x_1 + k^2 x_2}} = b(x_1 + k^2 x_2) \quad . \quad . \quad (94)$$

---

[1] This is not strictly accurate, as has been shown in Chapter IV. The true value for OB when the above assumptions are made is

$$OB = \frac{e}{x_1 + \dfrac{k^2 x_2}{\lambda_2}}$$

while the true value of OA is $\dfrac{eb}{\lambda_1}$. It follows that

$$\sigma = \frac{b(k^2 x_2 + \lambda_2 x_1)}{\lambda_1 \lambda_2}$$

But this correction only changes the value of σ on the average 3 or 4 per cent., so that the above approximation is commercially justifiable.

and
$$\nu = \frac{\sigma}{1-\sigma} = \frac{b(x_1 + k^2x_2)}{1 - b(x_1 + k^2x_2)} \quad . \quad . \quad . \quad (95)$$

Calculation of C_0.—In our example the leakage factor that is given is σ.

Hence for this machine

$$0{\cdot}1 = 0{\cdot}01(x_1 + k^2x_2)$$

So that $\quad x_1 + k^2x_2 = 10$

Usually also x_1 may be taken $= k^2x_2$

so that each would be 5 ohms.

Now with E as axis of reference

$$\begin{aligned} E(g + jb)(r_1 - jx_1) &= (0{\cdot}32' + 2{\cdot}5'')(1{\cdot}3 - j5) \\ &= 12{\cdot}9' + 1{\cdot}66'' \end{aligned}$$

Since E is 250, the left-hand side of equation (93) is

$$E\{1 - (g + jb)(r_1 - jx_1)\} = 237{\cdot}1' - 1{\cdot}66''$$

the vector 237·1 being in phase with E.

The right-hand side is

$$C_s\left(\frac{100 - 1{\cdot}3n}{40 - n}\right) - jC_s10$$

With 5 per cent. slip the frequency of rotation n is 38.

Hence
$$\begin{aligned} 237{\cdot}1' - 1{\cdot}66'' &= C_s\frac{100 - 49{\cdot}4}{2} - jC_s\,.\,10 \\ &= C_s\frac{50{\cdot}6}{2} - jC_s\,.\,10 \\ &= C_s(25{\cdot}3 - j10) \end{aligned}$$

This result emphasizes in an unmistakable way the desirability of what we have called the "axis of reference" in vector equations. Without such an axis it might be inferred here that the imaginary part on one side of the equation = the imaginary part on the other side, *i.e.* that

$$1{\cdot}66 = 10C_s$$

which is only true if these two quantities are in phase. Here they are not necessarily in phase; all we know is that C_s lags behind the vector $237{\cdot}1' - 1{\cdot}66''$ by an angle whose tangent is $\frac{10}{25{\cdot}3}$, that is, by about 22°; and that the vector $237{\cdot}1' - 1{\cdot}66''$ leads E by an angle whose tangent is $\frac{1{\cdot}66}{237{\cdot}1}$ (about $\frac{1}{2}$°).

From the equation

$$237{\cdot}1' - 1{\cdot}66'' = C_s(25{\cdot}3 - j10)$$

it is seen that the right-hand side, though made up of a real and imaginary, is not in terms of the same vector as the left-hand side. Without this the latter might be mistaken for an ordinary complex number.

One method of solving is as follows:—

Write $$C_s = \frac{237{\cdot}1' - 1{\cdot}66''}{25{\cdot}3 - j10}$$

Then rationalize (p. 57)

$$\begin{aligned} C_s &= \frac{(237{\cdot}1' - 1{\cdot}66'')(25{\cdot}3 + j10)}{(25{\cdot}3 - j10)(25{\cdot}3 + j10)} \\ &= \frac{6017' + 2329''}{640 + 100} = \frac{6017' + 2329''}{740} \\ &= 8{\cdot}13' + 3{\cdot}15'' \end{aligned}$$

Thus C_s is made up of two components, viz. 8·13 amperes in phase with E, and 3·15 amperes lagging behind E by 90°. The tangent of the angle of lag of C_s with respect to E is thus seen to be $\frac{3{\cdot}15}{8{\cdot}13}$.

$$\begin{aligned} \text{Also the R.M.S. value } C_s &= \sqrt{(3{\cdot}15)^2 + (8{\cdot}13)^2} \\ &= 8{\cdot}7 \text{ amperes} \end{aligned}$$

We could, of course, have arrived at this last result directly from the original equation by squaring and adding e components on either side and extracting the respective uare roots. Thus

$$(C_s 25{\cdot}3)^2 + (10C_s)^2 = (237{\cdot}1)^2 + (1{\cdot}66)^2$$

that

$$C_s = 8{\cdot}7 \text{ amperes}$$

ut we are then left without any knowledge of the angle etween C_s and E.

Output.—From (93*a*)

$$\begin{aligned} \text{output of motor} &= C_s{}^2 k^2 r_2 \left(\frac{n}{\sim - n} \right) \\ &= (8{\cdot}7)^2 \times 1{\cdot}2 \times \tfrac{38}{2} \\ &= 1728 \text{ watts per phase} \\ &= 5184 \text{ watts total} \\ &= 6{\cdot}83 \text{ H.P.} \end{aligned}$$

hich includes friction and windage.

Stator Current and Power Factor.—The tangent of the ngle of lag of C_s with respect to E is as shown above

$$\tan^{-1} \frac{3{\cdot}15}{8{\cdot}13} = \text{about } 22^\circ$$

The angle of lag of C_0 with regard to E is

$$\begin{aligned} \tan^{-1} \frac{b}{g} &= \tan^{-1} \frac{2{\cdot}5}{0{\cdot}32} \\ &= 82^\circ\, 42' \end{aligned}$$

We have with E as axis of reference

$$\begin{aligned} C_0 &= 0{\cdot}32' + 2{\cdot}5'' \\ C_s &= 8{\cdot}13' + 3{\cdot}15'' \end{aligned}$$

By direct addition

$$\begin{aligned} C_0 + C_s &= 8{\cdot}47' + 5{\cdot}65'' \\ \cos \phi &= \frac{8{\cdot}47}{\sqrt{(8{\cdot}47)^2 + (5{\cdot}65)^2}} = 83\,\% \\ C = C_0 + C_s &= \text{stator current at this slip} \\ &= \sqrt{(8{\cdot}47)^2 + (5{\cdot}65)^2} \\ &= 10{\cdot}2 \text{ amperes (nearly)} \end{aligned}$$

For the actual circle diagram of this machine see **Fig. 110.**

Single-phase Induction Motor.—The single-phase induction motor lends itself, of course, to vectorial treatment, but the expressions which result are not sufficiently simple to be of everyday practical use unless certain approximations are allowed.

With ordinary squirrel-cage rotor the behaviour of the machine at or near synchronism is very like that of the corresponding two-phase machine. As, however, the second phase on the stator does not exist, the magnetizing current for this phase has to be provided by the rotor, and its effect is transferred by rotor reaction to the active phase, so that a corresponding current is called for from the stator windings. This is evident from the discussion on p. 123, where it is shown that an E.M.F. $= E_2\frac{n}{\sim}$ occurs along the axis CD, Fig. 52, setting up along that axis a corresponding magnetizing current. Now, single-phase motors are so constructed that the exciting admittance along this axis is the same as that along the AB axis, viz. $(g + jb)$, and, as just said, this rotor magnetizing current is transferred to the windings of the active stator phase.

Hence the stator magnetizing current is nearly

$$E\left(1 + \frac{n}{\sim}\right)(g + jb) \quad . \quad . \quad . \quad . \quad (96)$$

As a matter of fact, the magnetizing current carried by the rotor will itself be the cause of an impedance drop in the rotor. This component is, however, usually so small as to be negligible, so that the expression given in equation (96)

$$E\left(1 + \frac{n}{\sim}\right)(g + jb)$$

is quite near enough.

When considering the action of the machine near synchronism $\frac{n}{\sim} = 1$, so that the magnetizing current is

$$E.2(g + jb)$$

or double the current per phase in the two-phase case.

Since in the single-phase case the rotor E.M.F. is as shown, p. 123,

$$E_2\left(1 - \frac{n^2}{\sim^2}\right)$$

we have the component $E_2\frac{n^2}{\sim^2}$ taking the place of $E_2\frac{n}{\sim}$ in the polyphase case, and of external load PD in the case of a transformer. Thus, still neglecting the impedance drop due to that magnetizing current which is carried by the rotor, we have in place of the equation for the polyphase motor (equation (91*a*))

$$E_2 = C_2Z_4$$

$$\text{where } Z_4 = \left(\frac{\sim}{\sim - n}n_2 - jx_2\right)$$

the form

$$E_2 = C_2Z_5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$\text{where } Z_5 = \left(\frac{\sim^2}{\sim^2 - n^2}r_2 - jx_2\right)$$

or instead of

$$\text{rotor E.M.F.} = E_2\frac{\sim - n}{\sim}$$

we get

$$\text{rotor E.M.F.} = E_2\frac{\sim^2 - n^2}{\sim^2}$$

So that the rotor current is

$$C_2 = E_2 \div Z_5$$

$$= E_2 \div \left(\frac{\sim^2}{\sim^2 - n^2}r_2 - jx_2\right) \quad . \quad . \quad . \quad . \quad (98)$$

So that the machine at a slip $(\sim - n)$ acts precisely as a two-phase motor, except that the rotor current (and consequently also the stator current C_s) is increased in the ratio $\frac{\sim + n}{\sim} : 1$, this being the effect of running with one stator phase instead of two.

From this reasoning it is easy to see that

$$C_s = E_1 \div \left(\frac{\sim^2}{\sim^2 - n^2} k^2 r_2 - j k^2 x_2\right)$$

So that in place of equation (93) we have

$$\begin{aligned} E\left\{1 - \frac{\sim + n}{\sim}(g + jb)(r_1 - jx_1)\right\} & \\ = E_1 + C_s Z_1 & \\ = C_s\left(\frac{\sim^2}{\sim^2 - n^2} k^2 r_2 - j k^2 x_2\right) + C_s Z_1 & \\ = C_s\left(\frac{\sim^2}{\sim^2 - n^2} k^2 r_2 + r_1 - j k^2 x_2 - j x_1\right) & \quad . \quad . \quad (99) \end{aligned}$$

Whence

$$\text{total watts delivered to rotor} = c_s^2 \frac{\sim^2}{\sim^2 - n^2} k^2 r_2 \quad (100)$$

output of motor (including friction losses)

$$= c_s^2 \frac{n^2}{\sim^2 - n^2} k^2 r_2 \quad . \quad . \quad . \quad (101)$$

and

$$\text{fractional slip} = \frac{\frac{n}{n + \sim} c_s^2 k^2 r_2}{\frac{n}{n + \sim} c_s^2 k^2 r_2 + \text{output}} \quad (102)$$

Thus for a given slip and output the rotor loss is increased by the ratio $\frac{\sim + n}{n}$ compared with that of the polyphase motor.

Another point must be mentioned with regard to the single-phase motor. If instead of a squirrel-cage rotor a three-phase wound rotor be used, then calling Z_2 the impedance per phase of the rotor, it is evident that in the single-phase case at standstill the effective impedance will vary slightly with the position of the rotor, because there is no rotation of the field. On the average it will be $\frac{2Z_2}{3}$. At synchronism, however, it will be $\frac{Z_2}{3}$; at intermediate speeds its value is $\frac{Z_2}{3}\,\frac{2\sim - n}{\sim}$.

This consideration is of importance only over wide ranges of speed; at normal values of slip the secondary impedance may be taken as $\frac{Z_2}{3}$.

Example II.—As an example let us take the three-phase motor recently dealt with, and calculate the output again for a slip of 5 per cent. We will assume that the same rotor is used and the same stator, but that the turns corresponding to two phases of the latter are connected in series and used for the single-phase case, while the third is left unwound. With the same air-gap density the stator applied voltage will then be about 480 (instead of 2 × 250) volts on account of the altered coil spread.

The stator resistance becomes 2·6 ohms, instead of 1·3 ohms per phase.

The rotor equivalent resistance changes as k^2, *i.e.* becomes

$$1{\cdot}2 \times \left(\frac{2{\cdot}6}{1{\cdot}3}\right)^2 = 4{\cdot}8 \text{ ohms per phase}$$

The stator reactance becomes 10 ohms practically there are now two coils each of 5 ohms in series, and coefficient of self-induction per coil will hardly change. rotor equivalent reactance also changes as k^2, and bec 20 ohms per phase.

But as there are three phases in the rotor and these then are in parallel, we must take their joint effect, which is as explained approximately :—

Equivalent secondary resistance

$$= \frac{4{\cdot}8}{3} = 1{\cdot}6 \text{ ohms} = k^2 r_2 \text{ in equation (99)}$$

Equivalent secondary reactance

$$= \frac{20}{3} = 6{\cdot}6 \text{ ohms} = k^2x_2 \text{ in equation (99)}$$

Since the density is kept constant, and the voltage is raised to suit the altered turns, the magnetizing current remains practically as before; that is, the admittance will alter in accordance with the number of turns in series and the flux-coil distribution. We have therefore when stationary

$$C_0 = E(g + jb) = 0{\cdot}32' + 2{\cdot}5'' \text{ amps.}$$

At synchronism this is, as just explained, doubled, and at other speeds

$$C_0 = \left(1 + \frac{n}{\sim}\right)(0{\cdot}32' + 2{\cdot}5'')$$

At 5 per cent. slip it is

$$(1{\cdot}95)(0{\cdot}32' + 2{\cdot}5'') = 0{\cdot}624' + 4{\cdot}875''$$

Multiplying this current by $r_1 - jx_1$ to get the drop in the stator due to it, we have the voltage vector

$$C_0Z_1 = 50' + 6{\cdot}4''$$

Since $e = 480$ the left-hand side of equation (99) is with E as axis of reference

$$430' - 6{\cdot}4''$$

and the right-hand side is

$$C_s\left(\frac{\sim^2}{\sim^2 - n^2}k^2r_2 + r_1 - jk^2x_2 - jx_1\right)$$
$$= C_s(19 - j16{\cdot}6)$$

Hence $\qquad 430' - 6{\cdot}4'' = C_s(19 - j16{\cdot}6)$

Hence C_s lags behind $(430' - 6{\cdot}4'')$ by

$$\tan^{-1}\frac{16{\cdot}6}{19} = \text{about } 41^\circ$$

(430′ − 6·4″) leads E by

$$\tan^{-1}\frac{6{\cdot}4}{430} = \text{about } 1^\circ$$

Solving for c_s

$$c_s^2(19^2 + 16{\cdot}6^2) = 430^2 + 6{\cdot}4^2$$

$$c_s = 17 \text{ amps. (nearly)}$$

$$\text{output of motor} = c_s^2\frac{n^2}{\sim^2 - n^2}k^2r_2 = 4300 \text{ watts}$$

$$= 5{\cdot}8 \text{ H.P. (nearly)}$$

$$\text{rotor loss} = c_s^2k^2r_2 = 464 \text{ watts}$$

It is evident that if the original winding was intended about 10 amperes it cannot carry 17 amperes, *i.e.* the p and output must be reduced till the safe current is ched. So a three-phase motor wound for, and run on, gle-phase circuits usually runs with a much smaller slip, keep the rotor loss and stator current within reasonable its.

The above calculation does not give us the power-factor. find this we must work to a definite axis of reference. us, taking E as axis of reference again, we have

$$430' - 6{\cdot}4'' = C_s(19 - j16{\cdot}6)$$

r the vector whose value is 430 is in phase with E;

$$C_s = \frac{430' - 6{\cdot}4''}{19 - j16{\cdot}6}$$

Rationalizing as on p. 58

$$C_s = \frac{8286' + 7018''}{637} = 13' + 11''$$

herefrom c_s might have been obtained instead of by the ethod used above, as $\sqrt{(13)^2 + (11)^2} = 17$ amps. nearly.

Since the ′ indicates a vector along the axis of reference, in phase with E, C_s is made up of two components, 13 in phase with E and 11 90° behind E. Thus the tangent of the angle of lag of C_s with respect to E is

$$\frac{11}{13} = 0{\cdot}846$$

and the angle itself is 40° nearly.

To find the total primary current we have with E as axis of reference

$$\begin{aligned} C &= C_0 + C_s \\ &= (0{\cdot}624' + 4{\cdot}875'') + (13' + 11'') \\ &= 13{\cdot}62' + 15{\cdot}87'' \end{aligned}$$

The R.M.S. value of the total primary current is then

$$\begin{aligned} c &= \sqrt{(13{\cdot}62)^2 + (15{\cdot}87)^2} \\ &= 21{\cdot}8 \text{ amps.} \end{aligned}$$

and the angle of lag ϕ of C behind E is

$$\begin{aligned} \phi = \tan^{-1}\frac{15{\cdot}87}{13{\cdot}62} &= \tan^{-1} 1{\cdot}16 \\ &= 49^\circ \end{aligned}$$

and $$\cos\phi = 0{\cdot}65$$

For the same slip, then, comparing the single- and three-phase case :—

	Three-phase.	*Single-phase.*
Slip	5 %	5 %
1^y voltage	250 per phase	480
1^y current	10·2 amps. per phase	21·8 amps.
H.P.	6·83	5·8
Rotor loss	273 watts	464 watts
$\cos\phi$	83 %	65 %

This comparison is exceedingly instructive. It has already been pointed out that the motor windings could not stand this stator current. If we regard 273 watts as the maximum rotor loss allowable, then in the single-phase case

$$c_s^2 k^2 r_2 = 273 \text{ watts}$$
$$c_s = 13 \text{ amps.}$$

This at once reduces the output to about 4 H.P., while the power-factor rises to about 0·75, the slip meanwhile decreasing to about 3 per cent.

Thus, on a basis of equal size and equal rotor loss, the single-phase induction motor compares very badly with the polyphase motor; while on a basis of equal slip it compares very badly with the polyphase motor from the point of view of size, efficiency, and power-factor.

CHAPTER VII

ALTERNATING CURRENT COMMUTATOR MOTORS

The Series Motor.—This machine, the simplest of a large class of alternating-current motors, is in arrangement precisely the same as a direct-current series motor, except that the field magnet is always laminated throughout to obviate eddy currents, and that special coils are often arranged around the armature brush-axis to reduce the armature reactance, this reactance being chiefly due to what is termed in direct-current work the armature cross-field.

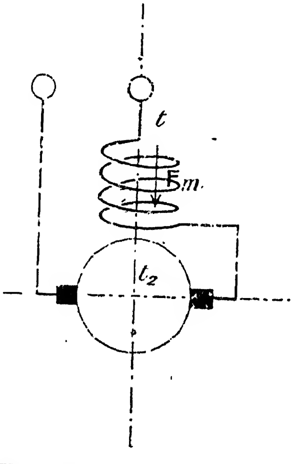

FIG. 56.—Diagrammatic series motor.

Fig. 56 illustrates in its simplest form the series motor, t standing for the number of turns in series on the field and t_2 for those on the armature, while the current through both is denoted by C.

Let x_2 be the reactance of the armature coils at ~ per sec. due to armature leakage flux (which includes the "cross-field").

Let r_2 be the armature and brush resistance.

Let x be the reactance of the coil t at ~ per sec. due to leakage flux.

Let r be the resistance of the coil t.

Then armature impedance at ~ per sec. $= Z_2 = r_2 - jx_2$, and impedance of coil t at ~ per sec. $= Z = r - jx$.

Let the impedance of the coil t due to the flux F_m which is common to the field and armature iron be

$$Z_m = r_m - jx_m$$

Then when the motor armature is stationary, the P.D. across the armature is CZ_2.

The P.D. across the coil t is always

$$C(Z_m + Z)$$

exactly as on p. 81.

The total P.D. at the terminals of the motor, so long as the armature is stationary, is

$$E = C(Z_2 + Z_m + Z) \quad . \quad . \quad . \quad (103)$$

Now, if the armature begin to rotate, the armature conductors cut the flux F_m. In so doing they will generate an E.M.F. whose maximum will occur at the same instant as that of F_m. To counterbalance this, then, we must add to the terminal P.D. an E.M.F. in phase with the flux F_m.

Now, CZ_m is 90° ahead of F_m (see Fig. 44). jCZ_m will therefore be an E.M.F. in phase with F_m.

But jCZ_m is proportional to the flux F_m, the frequency $\sim$ and the turns of the coil t.

The rotational E.M.F., on the other hand, must be proportional to the flux F_m, the frequency of rotation, and the turns of the armature t_2.

Let the armature rotate at n periods per sec., and let the ratio of turns on the field to turns on the armature be k.

Then if we divide the expression jCZ_m by the turns of the coil t and multiply it by the turns t_2, divide it also by $\sim$ and multiply it by n, we have the armature E.M.F. due to rotation which we require.

Thus E.M.F. counterbalancing that due to rotation

$$E_r = jCZ_m \frac{t_2}{t} \times \frac{n}{\sim} = jC\frac{Z_m}{k}\frac{n}{\sim}$$

The equation then for the P.D. at the motor terminals is obtained by adding this to the voltage of the motor when stationary as given in equation (103).

$$E = C\left(Z_2 + Z_m + Z + j\frac{Z_m}{k}\frac{n}{\sim}\right) \quad . \quad . \quad (104)$$

Now, if we separate all these impedances into their component parts, we get

$$E = C\left(r_2 + r_m + r + \frac{x_m}{k}\frac{n}{\sim}\right) - jC\left(x_2 + x_m + x - \frac{r_m}{k}\frac{n}{\sim}\right) \quad (105)$$

If E is constant the two expressions on the right-hand side form the sides of a right-angled triangle on a constant base, from which a circle diagram can be developed, as is shown on p. 228. Note here, however, that the term which contains the speed and r_m, the iron loss constant, tends to reduce x, *i.e.* to raise the power factor. The value of e is

$$e = c\sqrt{\left(r_2 + r_m + r + \frac{x_m}{k}\frac{n}{\sim}\right)^2 + \left(x_2 + x_m + x - \frac{r_m}{k}\frac{n}{\sim}\right)^2} \quad (106)$$

EXAMPLE I.—A single-phase series motor intended to work on a 25 ~ circuit has eight poles. The flux per pole at full load is 2×10^6 lines, and the number of field-turns is $2\frac{1}{2}$ per pole. The iron loss at full load is 1800 watts, the full-load current is 600 amps., the ratio of total armature-turns to total field-turns is 20, and the armature is parallel (lap) wound. If

$r = 0{\cdot}004$ ohm
$x = 0{\cdot}01$ ohm
$r_2 = 0{\cdot}003$ ohm
$x_2 = 0{\cdot}02$ ohm

find the terminal E.M.F., and the power factor at full load, with a speed of 600 R.P.M.

Note first that although the eight field poles are in series, the eight armature circuits are in parallel. The effective ratio of turns armature to field is therefore not 20, but $\frac{20}{8} = 2\frac{1}{2}$. So

$$k = \frac{1}{2{\cdot}5}$$

Value of Z_m.—The real value of this is given by the equation

$$cz_m = e_f$$

where e_f is the voltage across the field corresponding to the flux F_m.

$$\begin{aligned}\text{Now } e_f &= 4{\cdot}44 \times \text{turns in series} \times \text{flux per pole} \times \sim \times 10^{-8}\\ &= 4{\cdot}44 \times 20 \times 2 \times 10^6 \times 25 \times 10^{-8}\\ &= 44{\cdot}4 \text{ volts}\\ z_m &= \frac{44{\cdot}4}{600} = 0{\cdot}074 \text{ ohm}\\ Z_m &= r_m - jx_m\end{aligned}$$

where r_m is such a factor that

$$c^2r_m = \text{the iron losses} = 1800 \text{ watts}$$

Hence $$r_m = \frac{1800}{(600)^2} = 0{\cdot}005$$

We have therefore

$$\sqrt{r_m^2 + x_m^2} = 0{\cdot}074$$

and $$r_m = 0{\cdot}005$$

whence $$x_m = 0{\cdot}073$$

and (see Fig. 44) $\tan \alpha = \frac{r_m}{x_m} = \frac{0{\cdot}005}{0{\cdot}073}$; so α is less than $\frac{1}{2}°$

This is instructive, as showing that, even with large iron losses, the angle α is not very large and $\sqrt{r_m^2 + x_m^2}$ is approximately x_m.

Terminal E.M.F.—The component of terminal E.M.F. in phase with the current is from equation (105)

$$c.\left(r_2 + r_m + r + \frac{x_m}{k}\,\frac{n}{\sim}\right)$$

Now, n is in *cycles* per second, hence

$$n = \frac{600}{60} \times \frac{8}{2} = 40$$

so $$\frac{x_m}{k}\,\frac{n}{\sim} = 2{\cdot}5 \times 0{\cdot}073 \times \frac{40}{25} = 0{\cdot}292$$

This component of terminal E.M.F. then is

$$\begin{aligned} &600(0{\cdot}003 + 0{\cdot}005 + 0{\cdot}004 + 0{\cdot}292) \\ &= 600(0{\cdot}304) = 182 \text{ volts} \end{aligned}$$

The component of E.M.F. at right angles to the current is

$$c\left(x_2 + x_m + x - \frac{r_m}{k}\,\frac{n}{\sim}\right)$$

Now $$\frac{r_m}{k}\,\frac{n}{\sim} = 2{\cdot}5 \times 0{\cdot}005 \times \frac{40}{25} = 0{\cdot}02$$

So this component is

$$600(0{\cdot}02 + 0{\cdot}073 + 0{\cdot}01 - 0{\cdot}02) = 50 \text{ volts nearly}$$

The terminal E.M.F. is then

$$\mathrm{E} = \sqrt{(182)^2 + (50)^2} = 188{\cdot}8 \text{ volts}$$

and $\cos\phi$, the power factor, is $\dfrac{182}{188{\cdot}8} = 0{\cdot}96$

Repulsion Motor.—This machine has an armature like that of a direct-current motor, with a commutator, and brushes which are short-circuited. This armature is placed in a stator, furnished with a set of coils producing an appropriate number of poles. The short-circuited brushes are moved so that the resultant armature coil is turned through some small angle with respect to the field coils. Thus the bipolar form is diagrammatically illustrated in Fig. 57. Evidently the turns t of the stator coils can be considered as resolved into two parts, in series as shown dotted at t_a

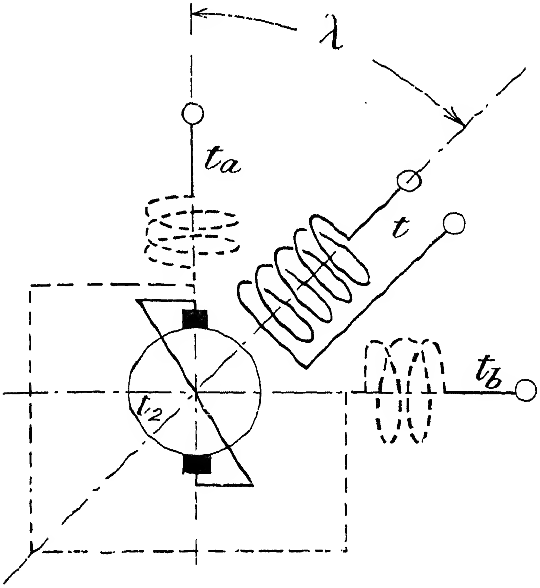

FIG. 57.—Diagrammatic repulsion motor.

and t_b, so that t_a and the short-circuited armature have a common axis, perpendicular to the axis of t_b.

$$t_a = t \cos \lambda$$
$$t_b = t \sin \lambda = t_a \tan \lambda$$

Tan λ is then the ratio of turns $t_b : t_a$.

t_a and the short-circuited armature behave exactly as transformer primary and secondary.

t_b behaves as a choking coil or series field coil; the flux from it enables the armature by rotation to set up an E.M.F. across the brushes.

The current through t_a and t_b is C_1, and we shall calculate the P.D. across the terminals of the coil t in terms of this current, exactly as we did for the series motor.

Evidently the impedance of the stator coils (due to their leakage flux and resistance), whether considered as a single coil (t) or as two coils in series ($t_a + t_b$), must be the same, so that as the first component of our terminal E.M.F. we get C_1Z, where Z is the impedance of the stator coils due to leakage flux at the frequency ~, and to their resistance.

We must now consider the relationships between t_a, t_b, and the armature turns t_2.

Let the ratio t ÷ armature turns $= k$
then t_a ÷ armature turns $= k \cos \lambda$
and t_b ÷ armature turns $= k \sin \lambda$

The machine may be considered simply as a series motor whose armature is supplied with current by *induction* from the coil t_a instead of by *conduction* from the coil t_b. Thus the equations will be very like those of the series motor, and the only real difference lies in the fact of the ampere-turns of the armature being less than the ampere-turns of the coil t_a, by the ampere-turns necessary to produce in the coil t_a, the air-gap, and the armature, their mutual flux.

Looking at the machine from this point of view, the equation for the E.M.F. in terms of the current may be written down direct. We have but to remember—

I. That we must consider the rotor current and E.M.F. as represented in, and reduced to terms of, the coil t_a.

II. That the rotor current is C_2 instead of C, and that it is represented in t_a by C_s, so that $C_s t_a = - C_2 t_2$, exactly as in the transformer.

III. That the rotor impedance is Z_2; so that the rotor impedance E.M.F. is C_2Z_2, which, when reduced to terms of the coil t_a, becomes

$$C_s \frac{t_a{}^2}{t_2{}^2} Z_2 \text{ or } C_1\left(\frac{t_a{}^2}{t_2{}^2} Z_2 \frac{C_s}{C_1}\right) \text{ or } C_1\left(k^2 \cos^2 \lambda Z \frac{C_s}{C_1}\right)$$

IV. That instead of the field coil t in the series motor, we now have the coil t_b; hence Z_m refers to this coil t_b, and k in the series motor here becomes $k \sin\lambda$.

V. That the E.M.F. to be applied to the rotor to counterbalance that set up by rotation must also be reduced to terms of the coil t_a; so we get for this E.M.F.

$$E_r = jC_1 \frac{Z_m}{k \sin\lambda} \frac{n}{\sim} \times \frac{t_a}{t_2} = jC_1 Z_m \frac{n}{\sim} \cot\lambda$$

Contrasting the two cases:—

Series Motor.

$$E = C\left(Z + Z_2 + Z_m + j\frac{Z_m}{k}\frac{n}{\sim}\right) \text{ (equation (104), p. 148)}$$

Repulsion Motor.

$$E = C_1\left(Z + k^2 \cos^2\lambda Z_2 \frac{C_s}{C_1} + Z_m + jZ_m \frac{n}{\sim} \cot\lambda\right) \quad (107)$$

From which it appears that the two machines are practically alike, if λ is constant, and C_s does not greatly differ from C_1.

Now, reference to p. 110 will show that F_m, the flux common to t_a, and the rotor sets up both the rotor E.M.F. and a corresponding E.M.F. in t_a.

The rotor E.M.F.'s C_2Z_2 and E_r, as represented in t_a, are—

$$C_s k^2 \cos^2\lambda Z_2 + jC_1 Z_m \frac{n}{\sim} \cot\lambda$$

The corresponding E.M.F. in t_a is also

$$C_0 Z_m \times \frac{t_a^2}{t_b^2} = C_0 Z_m \cot^2\lambda \quad . \quad . \quad . \quad (107a)$$

where C_0 is the magnetizing current for the flux common to t_a and t_2, exactly as in a transformer (p. 110).

For while the air-gap and magnetic circuit of the coil t_a must be considered as in all respects similar to that of the coil t_b, yet Z_m refers to the coil t_b, and here has to be reduced to terms of the coil t_a.

So $C_0 \cot^2 \lambda Z_m = C_1\left(k^2 \cos^2 \lambda Z_2 \frac{C_s}{C_1} + jZ_m \frac{n}{\sim} \cot \lambda\right)$

But $\quad C_0 = (C_1 - C_s)$

Substituting and rearranging, we get the relationship

$$\frac{C_s}{C_1} = \frac{Z_m\left(\cot^2 \lambda - j \frac{n}{\sim} \cot \lambda\right)}{k^2 \cos^2 \lambda Z_2 + \cot^2 \lambda Z_m} \quad . \quad . \quad . \quad (108)$$

and since in most cases $k^2 \cos^2 \lambda Z_2$ is small compared with $\cot^2 \lambda Z_m$, we may write

$$\frac{C_s}{C_1} = 1 - j \frac{n}{\sim} \tan \lambda = 1 - j \frac{t_b}{t_a} \frac{n}{\sim} \quad . \quad (108a)$$

so that C_0 is very nearly

$$= jC_1 \frac{t_b}{t_a} \frac{n}{\sim} \left(\text{or } jC_1 \frac{n}{\sim} \tan \lambda\right) . \quad . \quad . \quad (109)$$

i.e. for constant current and constant λ, C_0 varies as n.

The expression for $\frac{C_s}{C_1}$ may be put into general equation (107). But there is little object in doing this. It is better to keep equation (107) simple as it is, so as not to obscure its true meaning, and to work out the constant $1 - \frac{n}{\sim} \tan \lambda$ for each speed, or brush angle,[1] as is done in Example III., p. 157.

---

[1] As already stated, in this general equation Z_m refers to the coil t_b. As t_b and t_a are frequently united to form one coil t, it is convenient to replace Z_m by the corresponding quantity for the coil t. This will be obviously

$$\frac{t^2}{t_b^2} Z_m$$

and we will call it Z'_m. So that

$$Z'_m = \frac{t^2}{t_b^2} Z_m = \frac{1}{\sin^2 \lambda} Z_m$$

Then equation (107) becomes

$$\mathbf{E} = C_1\left(Z + k^2 \cos^2 \lambda Z_2 \frac{C_s}{C_1} + Z'_m \sin^2 \lambda + jZ'_m \frac{n}{\sim} \sin \lambda \cos \lambda\right) . \ (110)$$

EXAMPLE II.—*Show the relationship between open-circuit and short-circuit currents in the repulsion motor, and deduce a simple equation for the motor therefrom.*

Let us now suppose that the rotor is open-circuited, and that the full pressure E is applied to the stator terminals. Then write in equation (107) for the expression

$$C_s(k^2 \cos^2 \lambda Z_2) + jC_1 Z_m \frac{n}{\sim} \cot \lambda$$

its equivalent from equation (107*a*)

$$(C_1 - C_s) \cot^2 \lambda Z_m$$

so that

$$E = (C_1 - C_s)Z_m \cot^2 \lambda + C_1 Z + C_1 Z_m \quad . \; (111)$$

Let C_0 be the current flowing when the rotor is open-circuited. Then

$$E = C_0 Z_m \cot^2 \lambda + C_0 Z + C_0 Z_m \quad . \; . \; . \; (112)$$

for $\quad C_s = 0$

Thus

$$E = C_0 Z_m(1 + \cot^2 \lambda) + C_0 Z$$

$$= C_0\left(\frac{Z_m}{\sin^2 \lambda} + Z\right) = C_0(Z'_m + Z) \quad . \; . \; (113)^1$$

Next suppose that the angle λ is fixed at some reasonable value (less than 45°), and that the rotor is locked and short-circuited. Let the stator current C_x flow when the pressure E is again applied. Then

$$C_x = C_s \text{ practically}$$

for

$$C_s = C_x\left(1 - j\frac{n}{\sim} \tan \lambda\right)$$

and

$$n = 0$$

So from equation (107)

$$E = C_x(Z + k^2 \cos^2 \lambda Z_2 + Z_m)$$

Hence

$$\frac{C_0}{C_x} = \frac{Z + Z_m + k^2 \cos^2 \lambda Z_2}{Z + \dfrac{Z_m}{\sin^2 \lambda}} \quad . \; . \; . \; . \; (114)$$

---

[1] See footnote, p. 154.

If, as in the case of the induction motor, we neglect the resistance and iron losses (p. 134), then

$$\frac{c_0}{c_x} = \frac{x + x_m + k^2 \cos^2 \lambda x_2}{x + \dfrac{x_m}{\sin^2 \lambda}} = \sigma \quad . \quad . \quad . \quad (115)$$

This expression has a perfectly clear meaning when we remember that

$$\frac{x_m}{\sin^2 \lambda} = x_m \frac{t^2}{t_b{}^2}$$

i.e. it is the total reactance of the whole stator coil t due to flux common to it and the rotor; and in conjunction with x it may be measured easily, as shown, by determining the short-circuit current.

Thus an approximate equation for the repulsion motor is obtained by substitution in (107)

$$\mathrm{E} = \mathrm{C}_1 \left\{ - j\sigma\left(x + \frac{x_m}{\sin^2 \lambda}\right) + x_m \frac{n}{\sim} \cot \lambda \right\} . \quad (116)$$

which may be expressed by a simple semicircle diagram, when λ and E are constant, and n varies.

Example III.—The armature used for the single-phase series motor in Example I. is placed in a suitable stator and run as a repulsion motor. Assuming that the armature constants remain as before, that t_b is similar to the original series field coil t, and that coils t_a consist of 7·5 turns each, find the terminal E.M.F., and power-factor at the full load stator current of 600 amps., when the ~ of supply is 25, and the speed of the machine is synchronous, given that

$$r = 0{\cdot}012 \text{ ohm}$$

and $$x = 0{\cdot}04 \text{ ohm}$$

Since the speed is synchronous, $n = \sim$,

$$\tan\lambda = \frac{2\cdot5}{7\cdot5} = \frac{1}{3}$$
$$\cot\lambda = 3,\ \cot^2\lambda = 9$$
$$k\cos\lambda = \frac{t_a}{t_2} = \frac{6}{5} = 1\cdot2$$
$$k^2\cos^2\lambda = 1\cdot44$$

Value of Z_m *and* Z_2.—As t_b is similar to the original series field coil, we have

$$Z_m = r_m - jx_m = 0\cdot005 - j0\cdot073$$

and $$Z_2 = r_2 - jx_2 = 0\cdot003 - j0\cdot02$$

Relationship of Rotor and Stator Current.—From (108)

$$\frac{C_s}{C_1} = \frac{(0\cdot005 - j0\cdot073)(9 - j3)}{1\cdot44(0\cdot003 - j0\cdot02) + 9(0\cdot005 - j0\cdot073)}$$
$$= \frac{-0\cdot174 + j0\cdot672}{0\cdot049 - j0\cdot686}$$

The meaning of this is very obscure as it stands. Rationalizing, however, we get

$$\frac{C_s}{C_1} = \frac{0\cdot45 - j0\cdot19}{0\cdot47}$$
$$C_s = (0\cdot96 - j0\cdot4)C_1$$

or taking C_1 as axis of reference

$$C_s = 0\cdot96c_1' - 0\cdot4c_1''$$

So C_s leads C_1 by $\tan^{-1}\dfrac{0\cdot4}{0\cdot96}$ (about 22°).

If we use the approximation given above, equation (108*a*), viz.

$$\frac{C_s}{C_1} = 1 - j\frac{n}{\sim}\tan\lambda$$

we get $$C_s = c_1' - 0\cdot3c_1''$$

which would for most purposes be sufficiently near.

Value of the Rotor Current.—The rotor current is

$$C_s \times \frac{t_a}{t_2} = C_s \times k\cos\lambda = 1{\cdot}2C_s \text{ (p. 152)}$$

Now
$$\begin{aligned} C_s &= C_1(0{\cdot}96 - j0{\cdot}4) \\ &= (600 \times 0{\cdot}96)' - (0{\cdot}4 \times 600)'' \\ &= 576' - 240'' \end{aligned}$$

so
$$c_s = 624 \text{ amps.}$$

And the rotor current is $624 \times 1{\cdot}2 = 753$ amps.

Value of E *the applied Volts.*—From the above ratio we calculate the apparent rotor impedance, for inserting in equation (107)

$$\begin{aligned} k^2\cos^2\lambda Z_2\frac{C_s}{C_1} &= 1{\cdot}44(0{\cdot}003 - j0{\cdot}02)(0{\cdot}96 - j0{\cdot}4) \\ &= 1{\cdot}44(-0{\cdot}005 - j0{\cdot}02) \\ &= -(0{\cdot}0072 + j0{\cdot}0288) \end{aligned}$$

Also
$$\begin{aligned} Z &= r - jx = 0{\cdot}012 - j0{\cdot}04 \\ Z_m &= r_m - jx_m = 0{\cdot}005 - j0{\cdot}073 \\ jZ_m\frac{n}{\sim}\cot\lambda &= (0{\cdot}073 + j0{\cdot}005)3 = 0{\cdot}22 + j0{\cdot}015 \end{aligned}$$

Adding these together, their total is

$$Z + k^2\cos^2\lambda Z_2\frac{C_s}{C_1} + Z_m + jZ_m\frac{n}{\sim}\cot\lambda = (0\ 23 - j0{\cdot}127)$$

Whence from equation (107)

$$E = C_1(0{\cdot}23 - j0{\cdot}127)$$

Using C_1 again as axis of reference

$$\begin{aligned} E &= 0{\cdot}23c_1' - 0{\cdot}127c_1'' \\ &= (0{\cdot}23 \times 600)' - (0{\cdot}127 \times 600)'' \\ &= (138)' - (76)'' \end{aligned}$$

So **E** leads C_1 by $\tan^{-1}\frac{76}{138}$ (about 29°)

$$e = \sqrt{(138)^2 + (76)^2} = 157 \text{ volts}$$

and the power factor $= \cos\phi = \frac{138}{157} = 0{\cdot}88$

Result given by the approximate Equation.—It is interesting to compare with this the result given by using the very simple equations (115) and (116) given on p. 156:—

$$E = C_1\left\{-j\sigma\left(x + \frac{x_m}{\sin^2\lambda}\right) + x_m \frac{n}{\sim} \cot\lambda\right\}$$

$$\sigma = \frac{x + x_m + k^2\cos^2\lambda x_2}{x + \dfrac{x_m}{\sin^2\lambda}}$$

In our example

$$x + x_m + k^2\cos^2\lambda x_2 = 0{\cdot}04 + 0{\cdot}073 + 0{\cdot}0288 = 0{\cdot}142$$

$$x + \frac{x_m}{\sin^2\lambda} = 0{\cdot}04 + \frac{0{\cdot}073}{0{\cdot}096} = 0{\cdot}04 + 0{\cdot}76 = 0{\cdot}8$$

$$\sigma = \frac{0{\cdot}142}{0{\cdot}8} = 0{\cdot}177$$

Hence $$\sigma\left(x + \frac{x_m}{\sin^2\lambda}\right) = 0{\cdot}177 \times 0{\cdot}8 = 0{\cdot}142$$

and $$x_m \frac{n}{\sim} \cot\lambda = 0{\cdot}073 \times 1 \times 3 = 0{\cdot}22$$

So $E = C_1(0{\cdot}22 - j0{\cdot}142)$ by this approximation as compared with $E = C_1(0{\cdot}23 - j0{\cdot}127)$ accurately.

This gives $e = 156{\cdot}8$ volts
and $\cos\phi =$ nearly 88 per cent.

Use of a Single Coil.—Each pair of coils t_a and t_b can be united to form one coil t, as already shown. Since in the present instance each pole has

$$t_a = 7{\cdot}5 \qquad t_b = 2{\cdot}5$$

the coil t to replace these will consist of

$$\sqrt{(7{\cdot}5)^2 + (2{\cdot}5)^2} \text{ turns, } i.e. \text{ of } 7{\cdot}9 \text{ turns}$$

In practice, of course, 8 turns per pole would be used, and the brushes would be moved through an angle λ such that $\tan\lambda = \frac{t_a}{t_b} = 0{\cdot}33$; or $\lambda = 18^\circ\ 18'$ (Fig. 57).

The phase relationships in this machine at this load are given in the accompanying diagram.

By doubling t_a and t_b the rotor current will not be changed, nor will the angle λ, but C_1 will be halved and E about doubled. Thus the repulsion motor may be wound for any voltage, without the use of high pressure on the commutator. It is interesting to compare this example throughout with the corresponding case given under the series motor.

The example just given, and Fig. 58, bring out very clearly the curious phase relationships in the repulsion motor. For notice that C_s is *greater than* C_1, and *leads* it by a considerable angle. Similarly it is seen from the further calculation (p. 183) that E_1 *leads* E by a considerable angle. These conditions are quite different to those which exist in the transformer and induction motor (pp. 108, 136); and it is most interesting to compare the results, which is the more easily done since the letters used in all cases correspond.

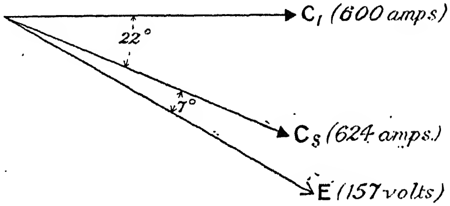

FIG. 58.—Phase relationships in the repulsion motor.

Compensated Motors.—If the series motor (Fig. 56) has a second winding on the armature connected down to another commutator whose brushes are placed in series with, but at right angles to, the first pair, we may consider the second pair of brushes with their winding as replacing the field coil t. If the two armature windings have each t_2 turns, then the equation for the series motor changes from (104) given on p. 148

$$E = C\left(Z + Z_2 + Z_m + j\frac{Z_m}{k}\frac{n}{\sim}\right)$$

to

$$E = C\left(2Z_2 + Z_m + jZ_m\frac{n}{\sim}\right) \quad . \quad . \quad . \quad (117)$$

The equation for this new machine is correct as far as it goes; but an effect ensues consequent upon putting the field winding on the armature which is, of course, not present in the series motor. This effect is the generation of an E.M.F. in the turns between the brushes of the field, due to rotation of these turns in any field which the other armature-turns set up along the axis of their brushes.

Now, the E.M.F. applied across the original brushes to balance that due to the armature self-induced field (or leakage flux) we have called $-jCx_2$. So that E.M.F. required to balance the rotational E.M.F. due to the same flux will be

$$j(-jCx_2)\frac{n}{\sim} = +Cx_2\frac{n}{\sim}$$

The final expression, then, for the E.M.F. of this machine is

$$E = C\left(2Z_2 + Z_m + jZ_m\frac{n}{\sim} + x_2\frac{n}{\sim}\right) \quad . \quad . \quad (118)$$

the new rotational E.M.F. $Cx_2\frac{n}{\sim}$ being exactly like that required to overcome an extra resistance. If instead of two armature-windings, one winding is used with two sets of brushes on the commutator at right angles, then the connection between them simply short circuits one section of the armature. The machine then works exactly as before, and the equations remain the same.

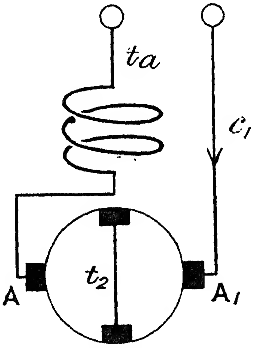

FIG. 59.—Diagram of elementary compensated motor.

Let us now turn to the form of this machine, which may be similarly developed from the repulsion motor; that is, let us (instead of providing a field from the coil t_b) place another pair of brushes on the commutator and, leading the main current through them, allow the resulting armature ampere-turns to take the place of the coil t_b. The diagrammatic form of such a machine is shown in Fig. 59.

Since the armature-turns now take the place of the coil t_b in Fig. 57, we have

$$\frac{t_a}{t_2} \text{ instead of } \frac{t_a}{t_b}$$

Let us call this ratio q. So

$$\frac{t_a}{t_2} = q$$

And $\quad \frac{t_b}{t_2} = \frac{t_2}{t_2} = 1 (= k \sin\lambda$ in the repulsion motor)

also $\quad \frac{t_a}{t_b} = \frac{t_a}{t_2} = q (= \cot\lambda$ and $k \cos\lambda$ respectively in the repulsion motor)

also Z, which before was the impedance of t_a and t_b, now is replaced by Z_a, which refers to t_a and the armature in series; while Z_m, which before referred to coil t_b, now refers to the armature-turns between A and A_1.

General Equation for the Compensated Motor.—The machine is precisely the same as the repulsion motor except for these changes in ratios of turns, and except for the E.M.F. introduced into the circuit between the brushes AA_1 due to the rotation of the armature-turns in the field common to t_a and the armature.

In the repulsion-motor equation (107)

$$E = C_1\left(Z + k^2 \cos^2\lambda Z_2 \frac{C_s}{C_1} + Z_m + jZ_m \frac{n}{\sim} \cot\lambda\right)$$

first substitute for the changed ratios, thus

$$E = C_1\left(Z_a + q^2 Z_2 \frac{C_s}{C_1} + Z_m + jqZ_m \frac{n}{\sim}\right) \quad . \quad . \quad . \quad (119)$$

The value of the new rotational E.M.F. in the field circuit AA_1 is arrived at as follows: t_a and the armature short circuit have a common flux which, oscillating through t_a, calls for an applied E.M.F. $C_0 q^2 Z_m$, or

$$C_1\left(q^2 Z_2 \frac{C_s}{C_1} + jqZ_m \frac{n}{-}\right) (cf.\ 107a)$$

It is the same flux which produces in the short-circuited rotor the total E.M.F.

$$C_s q Z_2 + jC_1 Z_m \frac{n}{\sim}$$

The E.M.F. to balance that produced by rotation in this same flux will be

$$j\left(C_s q Z_2 + jC_1 Z_m \frac{n}{\sim}\right)\frac{n}{\sim}$$

$$= -C_1 Z_m \frac{n^2}{\sim^2} + jqC_s Z_2 \frac{n}{\sim}$$

Putting this into the previous equation, we have as the general equation for compensated motors

$$\begin{aligned} E = C_1\Big\{Z_a + \frac{C_s}{C_1}\left(q^2 Z_2 + jqZ_2 \frac{n}{\sim}\right) \\ + Z_m\left(1 - \frac{n^2}{\sim^2}\right) + jqZ_m \frac{n}{\sim}\Big\} \quad . \quad . \quad . \quad (120)\end{aligned}$$

Compensation.—At synchronism $n = \sim$.

Then the term $Z_m\left(1 - \frac{n^2}{\sim^2}\right)$ disappears completely, and the largest impedance in the machine, that due to the field flux, is *compensated.* As a result, the power factor is much improved, and at a higher speed still more compensation takes place. For simplicity's sake assume for the moment that $C_s = C_1$. Then we have the component voltages—

In phase with C_1.	*At right angles to* C_1.	
(1) $C_1 r_a$	$-jC_1 x_a$	(coil t_a and armature)
(2) $C_1 q^2 r_2$	$-jC_1 q^2 x_2$	(armature)
(3) $C_1 q x_2 \frac{n}{\sim}$	$+jC_1 q r_2 \frac{n}{\sim}$	(armature)
(4) $C_1 r_m$	$-jC_1 x_m$	(armature field)
(5) $-C_1 r_m \frac{n^2}{\sim^2}$	$+jC_1 x_m \frac{n^2}{\sim^2}$	(armature field)
(6) $C_1 q x_m \frac{n}{\sim}$	$+jC_1 q r_m \frac{n}{\sim}$	(armature field)

The condition for unity power factor is that the sum of the above j terms shall be zero, *i.e.*

$$x_m\frac{n^2}{\sim^2} + \frac{n}{\sim}(qr_m + qr_2) = x_a + q^2x_2 + x_m \quad . \quad (121)$$

If x_a and q^2x_2 are reasonably small, as they should be, complete compensation takes place just above synchronism.

Simplification of the Equation.—We have shown that for ordinary purposes in the case of the repulsion motor it is justifiable to neglect certain quantities.

We cannot carry out the same simplification here with the same advantage except as regards the ratio $\frac{C_s}{C_1}$.

Ratio of $\frac{C_s}{C_1}$.—This is obtained in exactly the same way as for the repulsion motor, and gives (see equation (108))

$$\frac{C_s}{C_1} = \frac{q^2Z_m - jqZ_m\frac{n}{\sim}}{q^2Z_m + q^2Z_2} \quad . \quad . \quad . \quad . \quad (122)$$

Neglecting Z_2

$$\frac{C_s}{C_1} = \left(1 - j\frac{n}{q\sim}\right) \quad . \quad . \quad . \quad . \quad . \quad (123)$$

Hence $$C_0 = jC_1\frac{n}{q\sim} \text{ (nearly)}$$

Relationship of Open-circuit to Short-circuit Currents.—With rotor on open circuit, $C_s = 0$, $n = 0$. Hence

$$E = C_1(Z_a + q^2Z_m + Z_m)$$

On short circuit with rotor locked

$$C_1 = C_s \text{ (practically)}$$
$$E = C_1(Z_a + q^2Z_2 + Z_m)$$

If C_0 and C_x denote the open and short-circuit currents respectively for the same value of E, then

$$\frac{C_0}{C_x} = \frac{(Z_a + q^2Z_2 + Z_m)}{Z_a + q^2Z_m + Z_m} \quad . \quad . \quad . \quad . \quad (124)$$

Neglecting again the iron and resistance losses

$$\frac{\mathfrak{E}_0}{\mathfrak{E}_x} \; \frac{\mathfrak{E}_\theta}{\mathfrak{E}_x} = \frac{x_a + q^2 x_2 + x_m}{x_a + q^2 x_m + x_m} = \sigma \quad . \quad . \quad (124a)$$

This, however, is precisely similar to the expression for a repulsion motor (p. 156). Hence the compensated motor has practically no advantages over the repulsion motor at starting, *i.e.* when n is very small.

Simplest Form of Equation.—It is clear that at any speed approaching synchronism the part $C_1 r_m$ is practically compensated. On the other hand, the primary and secondary resistances ought to be very small. For most purposes, therefore, the equation

$$E = C_1\left\{-j\left(x_a + q^2 x_2 + x_m - \frac{n^2}{\sim^2} x_m\right) + q x_m \frac{n}{\sim}\right\} \; . \; (125)$$

is sufficient instead of the more accurate form (120), and this may be written

$$E = C_1\left\{-j\sigma(x_a + q^2 x_m + x_m) + j\frac{n^2}{\sim^2} x_m + q x_m \frac{n}{\sim}\right\} \; . \; (126)$$

In this motor the stator windings cannot be considered independently of the rotor windings, as is the case with the repulsion motor (p. 160), because the field turns are t_2. Thus q is a very important factor.

Adjustment of Compensation.—It follows from equation (121) that with a motor constructed on the lines of Fig. 59 complete compensation can only be secured at a speed just above synchronism, and further, that exciting from the armature takes away the convenience afforded by the variable angle λ in the repulsion motor.

To overcome these defects it is necessary to be able to vary the number of exciting-turns; but, as this number is decided by the number of armature-turns, it cannot be conveniently done on the armature itself. There is, however, a method available, for we may counteract some of the field-turns on the armature by a variable number of opposing turns on the stator. This arrangement is depicted in Fig. 60, where t_a and t_2 are as in Fig. 59. In series with t_a and opposing (or assisting) the ampere-turns set up by the armature along the axis AA_1 is a number of turns t_3 which can be varied. The change which takes place in equation (125) is easily traced. The impedance Z_m along the axis AA_1 is reduced in the ratio of $(t_2 - t_3)^2 : t_2^2$, but this reduction affects the ampere-turns acting along the axis of t_a which produce the compensation.

For, as a result of the substitution of the value $t_2 - t_3$ for the original turns t_2, the flux across the armature along

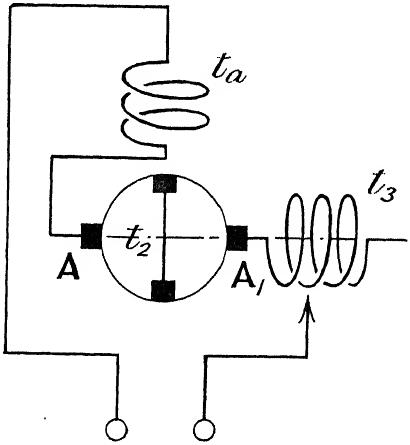

FIG. 60.—Motor with adjustable field-turns.

the axis AA_1 will be reduced in a corresponding manner. Consequently the rotational E.M.F. per revolution due to this flux will be similarly reduced, and this, as we know, corresponds to a decrease in C_0 and the flux which crosses the air-gap along the axis of the short-circuited brushes (see p. 164). It follows that the speed at which compensation takes place is only slightly changed, but that the current corresponding to this speed is really the factor most affected. The introduction of the variable turns t_3 then has an effect on the behaviour of the motor like changing the angle λ in the repulsion motor, and a similar effect may be produced by moving either set of brushes through some small angle from their normal position instead of by the addition of the turns t_3.

Compensated Constant Speed Motors.—If when such a motor as that shown in Fig. 59 attains a certain speed the potential difference across the brushes AA_1 is kept constant by placing the brushes in parallel across the line (either directly or through a transformer), the machine tends to run at almost constant speed like a shunt motor. The only condition for satisfactory operation is that the voltage applied across the brushes AA_1 shall be practically the same as that which exists there when the motor is running fully compensated. This type of machine is known as a "shunt compensated motor," and has one drawback, viz. that it will not start from rest without some special device or connection. That which is usually adopted is an arrangement whereby the machine is started as in Fig. 59, and afterwards at the proper speed the brushes AA_1 are put in parallel with the line as described. The equation for such a case is easily deduced from equation (125); for the only alteration is that C_1Z_m, instead of depending upon C_1, is constant at the value of C_1 corresponding to compensation, and the E.M.F.'s depending upon it are altered to correspond.

The student will find the resulting change in the equations an excellent example for practice. He can show that the approximate equation for the coil t_a is

$$E = qE\frac{n}{\sim} - jC_1(x_a + q^2x_2) \quad . \quad . \quad (127)$$

where x_a is the leakage reactance of the coil t_a. Thus if $x_a + q^2x_2$ is quite small (as it should be), the speed will be very nearly constant over a wide variation of the value of C_1

CHAPTER VIII

THE PRODUCT OF TWO VECTORS

BEFORE attempting any general definition of the products of vectors, it may be well to consider briefly one or two points regarding the meaning of products of physical quantities in general.

When we speak of the product of two physical quantities, we have to keep two things in mind, viz. the mathematical product and its physical interpretation.

The product in such a case must necessarily be different in its physical nature from either of the factors forming the product, *i.e.* it will be measured in different units. Thus, if we multiply a velocity by a period of time, we obtain a displacement as the result. Similarly, a force multiplied by a velocity gives a power as the product. In all cases of this kind it is impossible from the rules of pure mathematics to deduce any relation between the physical nature of the factors and of the product. Pure mathematics can only show the connection between the numerical values of factors and their product, *i.e.* it can only enable us to calculate the number of units contained in the product from the known number of units of each factor. It is the function of physics, and not of mathematics, to assign the physical meaning to the result of a multiplication, and to define the units in which factors and product must be measured. It is consequently necessary to remember that, although mathematically we may be able to obtain a product by the multiplication of any two numerical quantities together, the result may, or may not, have a physical meaning. In any case the physical meaning remains to be determined independently of the mathematical rules by which the multiplication is carried out.

For example, 10 × 15 = 150 shows a mathematical product. If the factors on the left of the equation are 10 houses and 15 trees, it would obviously be difficult to assign any physical meaning to the product on the right-hand side of the equation. If the factors were 10 amperes and 15 volts, we might state the product to be 150 watts. Mathematically, one case would be as simple as the other. Physically, we could only usefully employ the process of multiplication in the second case.

We must therefore consider separately the *numerical product* of the quantities which is independent of units, and the *product of the units* which gives rise to a resultant unit always different from the units of either component.

From this it follows that the practical use which can be made of the mathematical process of multiplication depends on the possibility of putting a useful physical interpretation on the product. In alternating-current problems we shall find that there are only a very few cases of ordinary occurrence in which use can be made of the multiplication together of two vectors. Such cases are: The multiplication together of currents and voltages, giving a product in units of power (watts); the multiplication of magnetic flux and current, giving a mechanical force (dynes) as the product, etc.

Geometrical Meaning of the Product of Two Vectors.—

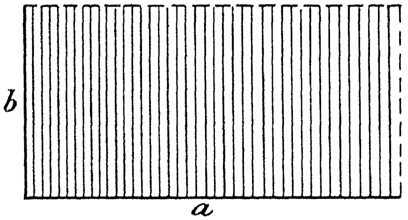

FIG. 61.—Geometrical product of two lines.

By the product of two lines in geometry we understand the area obtained by setting one line perpendicular to the other and imagining it to move parallel to itself through a distance equal to the length of the first line. Thus if a, b are two lines, the product $a \times b$ is the shaded area swept through by the line b when made to move parallel to itself through the distance a.

This geometrical product agrees with the algebraic product $a \times b$, since the number of units of area in the shaded portion of Fig. 61 is equal to the number of units of length of line a multiplied by the number of units of length of b. Herein lies the justification for representing an area in algebraic form as $a \times b$, and also the reason for attempting to express the product of two vectors A × B in the form of an area.

Vector Product.—In the case of vectors, we have to deal with lines which have definite inclinations as well as definite lengths. A method of forming the product which is similar to the geometrical method given above, may, however, be employed in this case.

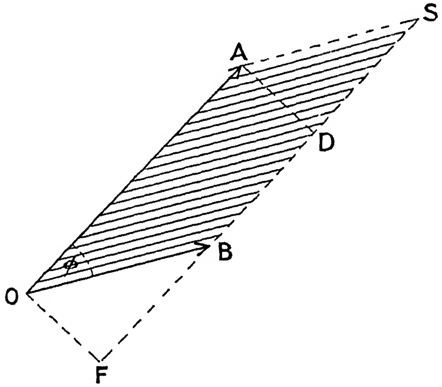

FIG. 62.—Vector product of two vectors.

Let A, B be two vectors enclosing an angle ϕ (see Fig. 62).

By sliding the vector B along the vector A parallel to itself, it will sweep through the shaded area OASB. This area is evidently equal to the rectangle ADFO described on OA.

The area of this rectangle is obviously equal to $ab \sin \phi$ units of area if a, b are the lengths of A and B respectively.

The numerical value of the product of two vectors of length a, b, and making an angle ϕ with one another when derived in this way, is equal to $ab \sin \phi$. It is to be noted that the angle ϕ must be measured according to some definite rule, since the two areas obtained by sliding B along A in Fig. 63 would be different for (*a*) and (*b*) (although numerically equal), notwithstanding that the lengths of the vectors and the included angle are the same in both cases. This distinction must be drawn by considering the sign of the angle between A and B. In Fig. 63 (*a*) the angle measured from A to B is positive (angles measured in a counter-clockwise direction we have assumed positive). In Fig. 63 (*b*) the angle measured from A to B is negative, and herein lies the distinction between the two cases. In vector multiplication we must write the product as negative if the sense of the

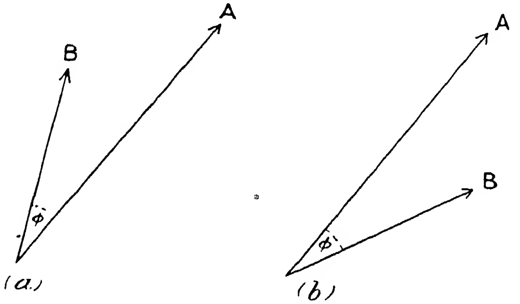

Fig. 63.—Vectors enclosing positive and negative angles.

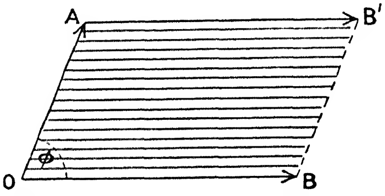

Fig. 64.—Negative product of vectors.

angle measured from the first vector to the second vector is negative, and *vice versâ*.

The sign of the product may also be obtained in another way. In obtaining the area which represents the product, the second vector is made to slide along the first vector. In the final position of the second vector, the arrow-heads of both vectors will be directed in the same sense round the area swept through by the second vector (see Fig. 64). If the direction of the vectors round the area is clockwise, the product is negative; if counter-clockwise, the product is positive. Thus in Fig. 64 the product A × B is negative. In Fig. 65 the product B × A is seen to be positive.

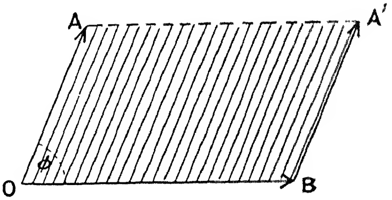

FIG. 65.—Positive product of vectors.

Definition.—The product of two vectors, obtained by sliding one vector parallel to itself along the other, is called the *vector product* of the vectors. Numerically the vector product of two vectors A, B of length a and b, and inclined to one another at the angle ϕ measured from A to B, is $ab \sin \phi$.

This product must itself be a vector quantity, since it has already been shown to possess a definite sense. Also, the area representing this product has a definite position relative to the two vectors.

A special case of immediate importance to us, in which the sense of the angle between a pair of vectors produces a change in the sign of their product, is the case when two quantities measured along mutually perpendicular axes are multiplied together.

For instance, if a' is multiplied by b'' (see Fig. 66), when both a' and b'' are positive, the result will be positive.

since the angle between the direction of a' and of b'' is 90° measured in a *positive* sense.

On the other hand, b'' multiplied by a' will be a negative product, since the angle between b'' and a' is 90° measured in a *negative* sense. Thus

$$a' \times b'' = -b'' \times a'$$

This is a case where the laws of algebra are not followed in the multiplication of vectors, and is expressed by saying that the vector product of two vectors is not commutative.

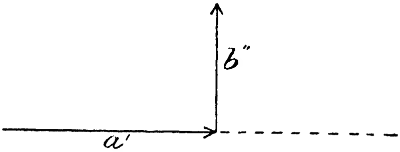

FIG. 66.—Product of two mutually perpendicular vectors.

In general it is true of vector products that

$$A \times B = -B \times A$$

since the sense of the angle is always to be measured from the first vector to the second.

The vector which comes first in the product is always the vector from which the angle between the vectors is measured. If the angle from the first to the second vector is positive, the product is positive, and *vice versâ*.

Scalar Product.—While still adopting the analogy of obtaining the product of two lines as an area by sliding one line relatively to the other, we may form an area representing the product of two vectors in another way from that described above under the heading "vector product."

Again taking two vectors A and B inclined to one another at an angle ϕ, let us imagine the second vector, B, as describing an area by moving parallel to itself in a direction *at right angles to the first vector*, instead of parallel to it. The resulting area is the shaded one in Fig. 67. This area is evidently equal to the area of the rectangle ODFS, *i.e.* to

$a \times b \times \cos \phi$, where a, b are the lengths of the vectors A, B.

The product of two vectors formed in this manner is called the *scalar product* of the vectors, since it differs in important respects from the vector product previously defined.

In the first place, this product is a scalar and not a vector quantity, since it is unaltered by a change in the sign of the angle between the vectors, for $\cos \phi = \cos (-\phi)$, so that it is immaterial in which direction the angle between the vectors is measured. In other words, this product is commutative, so that A × B = B × A. Also, the area swept through by the moving line OB (Fig. 67) does not lie on any definite plane, and has consequently no fixed position in space relative to the vectors. There is thus no definite sense connected with this product.

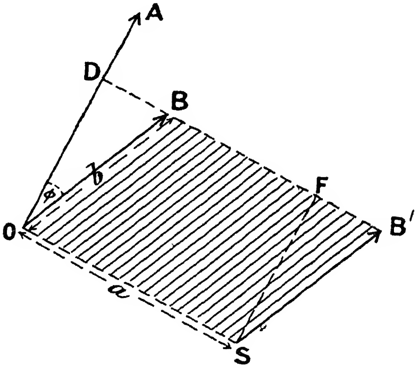

FIG. 67.—Scalar product of two vectors.

Definition.—The scalar product of two vectors of length a and b and inclined to one another at an angle ϕ is $ab \cos \phi$.

Vector and Scalar Products.—We can thus form the product of two vectors in either of two ways; and we obtain a different result in the two cases. Mathematically both products are possible; in physical problems one will usually give a real product, and the other an imaginary value for the product.

We have in practice, therefore, to employ either the scalar or the vector product, according to the physical nature of the vector quantities dealt with.

It is evident that the product of two vectors in phase with one another can have no vector component, since $\sin\phi = 0$. Similarly, the product of mutually perpendicular vectors can have no scalar component, because in this case $\cos\phi = 0$. Hence, by expressing all vectors in terms of component vectors along two perpendicular axes, as is done in symbolic notation, the vector and scalar parts of the resultant product are separated during the process of multiplication. This must be so, because each component vector is always multiplied by another component either parallel or perpendicular to itself.

It is evident from another consideration that the product of two parallel vectors must give a scalar quantity, because, since $\phi = 0$, the angle, which is the only factor giving a positive or negative sense to the product, has disappeared.

Product of Two Vectors in Symbolic Form.—If the two vectors are represented in our usual symbolic form

$$\begin{aligned} \mathrm{A} &= a_1{}' + a_2{}'' \\ \mathrm{B} &= b_1{}' + b_2{}'' \end{aligned}$$

their product obtained by the usual rules of algebra will consist of the following terms

$$a_1{}'b_1{}' \qquad a_2{}''b_2{}'' \qquad a_1{}'b_2{}'' \qquad a_2{}''b_1{}'$$

Each of the first two terms consists of the product of two parallel vectors, for the components of each term are lengths measured along a common axis. In fact, in forming these products the question of relative inclination does not arise. The terms $a_1{}'b_1{}'$ and $a_2{}''b_2{}''$ are therefore both pure scalar products, and are formed according to the ordinary rules of algebra. They are thus commutative, and it is immaterial whether we write $a_1'b_1'$ or $b_1'a_1'$. The sign to be prefixed to these terms will be the sign determined by algebraic rules. In the present case both terms will be positive, and we may write them

$$a_1'b_1' + a_2''b_2''$$

We can show in another way that these first two terms in the product of A and B represent the scalar product of the vectors A, B, as already defined.

For we may show trigonometrically that $a_1'b_1' + a_2''b_2''$ is equal to the value $ab \cos(\alpha - \beta)$ (see p. 30), since

$$\begin{aligned} a_1' &= a \cos \alpha \\ b_1' &= b \cos \beta \\ a_2'' &= a \sin \alpha \\ b_2'' &= b \sin \beta \end{aligned}$$

Inserting these values

$$\begin{aligned} a_1'b_1' + a_2''b_2'' &= ab(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= ab \cos(\alpha - \beta) \end{aligned}$$

This we have already defined as the scalar product of the vectors A, B.

The two remaining terms of the product as given at the beginning of this section, viz. $a_1'b_2''$ and $a_2''b_1'$, are both purely vector products, since each is the product of two vectors at right angles to one another. This fact separates them at once from the terms composing the scalar product. Although these vector terms $a_1'b_2''$ and $a_2''b_1'$ have a different physical meaning from the preceding scalar terms, it does not follow that they are without a useful signification. In fact, as previously stated in connection with "imaginary" vectors (see p. 44), we may conceive them to be measured along another axis (viz. an axis at right angles to the plane of the original vector diagram).

Each of these latter terms, then, is the product of two mutually perpendicular vectors, so that they do not, like the first two terms, come directly under the ordinary algebraic rules for multiplication in which the sense of direction is not considered. In fact, as already pointed out, each of the terms is a purely vector quantity, and its sign must accordingly be obtained by the rules for vector multiplication. In the present case, the rule given on p. 171 for the sign of the product of two perpendicular vectors must be applied in order to determine the signs to be prefixed to these terms.

In accordance with the rule, the angle between a_1' and b_2'' is positive, while the angle between a_2'' and b_1' is negative. We thus get the last two terms of the product of A × B as

$$a_1'b_2'' + a_2''b_1' \text{ equal } \textit{numerically} \text{ to } a_1b_2 - a_2b_1$$

As already pointed out, these terms have a different physical meaning from the first (scalar) terms, and the product which they represent is to be considered as to be measured along an axis perpendicular to that of the original vectors.

From the manner of obtaining the product, it is evidently not commutative (like the scalar product). Both of these points are characteristic of a *vector product*. We can now proceed to show that the numerical value of the vector terms

$$a_1'b_2'' + a_2''b_1'$$

agrees with the value previously given for the vector product of the vectors A, B.

By substituting trigonometrical values for these terms, we get

$$\begin{aligned} a_1b_2 - a_2b_1 &= a\cos\alpha \,.\, b\sin\beta - a\sin\alpha \,.\, b\cos\beta \\ &= ab\sin(\beta - \alpha) \end{aligned}$$

This agrees with the numerical value of the vector product as given on p. 172.

The full expression for the product of the vectors A, B may now be written

$$A \times B = a_1'b_1' + a_2''b_2'' + a_1'b_2'' + a_2''b_1' \quad . \quad (128)$$

We thus find that when the product of two vectors is obtained by multiplying together the usual symbolic form of the vectors by the usual rules of algebra (but with special regard to sign in the case of the terms containing mutually perpendicular elements, to take account of the sense of the angle between these elements) we obtain a product which is the sum of the scalar and vector products. Further, the scalar product is commutative, and has no definite line of action, while the vector product is non-commutative, and must be considered as acting along an axis perpendicular to the plane of the original vectors.

As already mentioned, in any given problem, usually only one of these products will have a real meaning, while the other must be considered to be imaginary. Which product is to be taken as significant in any particular case will depend on the physical units in which the vectors are measured.

Illustration of Product of Vectors from Curves plotted to Rectangular Co-ordinates.—When employed in connection with alternating problems, the relation of the scalar and vector products to the original vectors is shown in an interesting manner by plotting the quantities in the form of curves referred to rectangular co-ordinates.

The two curves marked A, B in Fig. 68 show two simple harmonic functions having maximum values a and b respectively, and differing in phase by 30°. By multiplying together the ordinates of the curves which correspond to the same instant of time, we obtain a curve showing the variation of the product of the functions. It is seen that this curve (marked P in Fig. 68) is a sine curve like the curves representing the original vector quantities. The curve P is further seen to have double the frequency of the original curves, and is displaced upwards, its axis being at a height indicated by the dotted line above the axis of A and B. It can be shown that the amplitude of the curve P depends only on the amplitudes of the original curves A and B, and is independent of the difference in phase between them. The height of the axis of curve P above that of A and B is determined by the difference of phase between A and B and by the amplitudes of the curves.

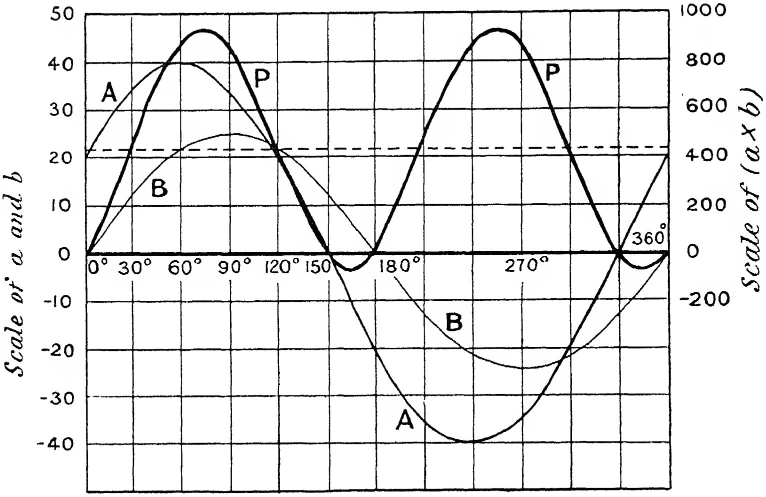

FIG. 68.—Product of vectors plotted to rectangular co-ordinates.

The form of the curves may be shown trigonometrically as follows:—

Let the curves A, B be represented by

$$A = a \cos \theta$$
$$B = b \cos (\theta + a)$$

a and b being the maximum values of the functions
θ being the varying angle $2\pi \sim t$
a being the constant angle of phase difference between the curves.

The product is then given as

$$\begin{aligned} P = A \times B &= ab \,.\cos \theta \,.\cos (\theta + a) \\ &= ab \cos \theta\,(\cos \theta \cos a - \sin \theta \sin a) \\ &= \frac{ab}{2}\{\cos a\,(1 + \cos 2\theta) - \sin a \sin 2\theta\} \\ &= \frac{ab}{2}\{\cos a + \cos (2\theta + a)\} \quad . \quad . \quad . \quad (129) \end{aligned}$$

This expression contains a constant term $\frac{1}{2}ab \cos a$, which represents the height of the axis of the curve P in Fig. 68. This is the *scalar product* of A and B (as previously defined), and cannot be represented by a rotating vector, since it is not an alternating quantity.

We thus arrive at the value of the *average power* in a circuit in which the original curves AB represent the current and applied voltage respectively, since power is a scalar quantity, having no direction. This term is in this case the only significant part of the product.

The second term of the product

$$\frac{ab}{2} \cdot \cos (2\theta + a)$$

forms the vector part of the product, and is seen to be an alternating quantity having double the frequency of the original vectors. It represents the curve of this character in Fig. 68.

This suggests a new meaning to the statement that the vector product has a special physical significance. The term "imaginary" sometimes applied to this product indicates in this case "of another frequency," thus showing that vectors of different frequency are not comparable.

It was previously stated that the vector product was sometimes to be considered to be "imaginary" in the sense of being measured along an axis perpendicular to the plane of the vector diagram.

When the scalar product

$$\frac{ab}{2} \cos a$$

represents the true power in a circuit, the meaning to be assigned to the vector product is that it represents the alternately positive and negative fluctuations of power in the circuit, which have a double frequency and a mean value of zero. This fluctuating power has been called by Steinmetz the "wattless volt-amperes," and represents the product of the idle current and voltage of the circuit. In terms of the applied voltage and current its numerical value has already been shown to be

$$\frac{ab}{2} \sin \alpha$$

Product of Two Vectors (Exponential Form).—When obtaining the product of two vectors expressed in the form

$$A = a\varepsilon^{j\theta}$$

and $$B = b\varepsilon^{j(\theta + \alpha)}$$

the fact that the multiplication is non-commutative in the case of vectors, and that the sense of the angle between the vectors must be taken into account, prevents the application of the usual algebraic operations, as employed for the product of a vector and a complex number.

Thus, although the form of factors producing the product is identical in the multiplication of one vector by another vector and by a complex number, the operation can only be carried out in the case of multiplication by a complex number, in which case the product is a simple vector, as explained on p. 55.

Examples of Vector Products.—The problem of obtaining the results from the multiplication of two vectors is complicated in actual practice by the presence of complex numbers, such as impedances, etc. If, however, the expressions to be multiplied together are first made to refer to a definite axis the difficulties become no greater than in the case of the product of the vectors A and B just discussed.

EXAMPLE I.—As a first instance we may calculate the input in watts to the transformer in the example on p. 111. It is there shown that the applied voltage is

$$E = 3019' - 29{\cdot}83''$$

with C_s as axis of reference.

In the same example the total primary current was

$$C_1 = C_0 + C^s = 1{\cdot}44' + 0{\cdot}17''$$

with C_s again as axis of reference.

Input therefore

$$= \mathrm{E}\,.\,\mathrm{C}_1 = (3019' - 29{\cdot}83'')(1{\cdot}44' + 0{\cdot}17'')$$

Regarding E and C_1 as the vectors A and B just generally discussed, and multiplying out accordingly

$$\mathrm{EC}_1 = e_1'c_1' + e_2''c_2'' + e_1'c_2'' + e_2''c_1'$$

i.e.

$$\mathrm{EC}_1 = (3019' \times 1{\cdot}44') - (29{\cdot}83'' \times 0{\cdot}17'') + (3019' \times 0{\cdot}17'') - (29{\cdot}83'' \times 1{\cdot}44')$$

The first two terms are the scalar product, and give 4342 watts practically. The other two terms are

$$-(0{\cdot}17'' \times 3019') - (29{\cdot}83'' \times 1{\cdot}44')$$

as just explained; they give the value

$$-514 - 43 = -547$$

wattless volt-amperes of double the normal frequency, which would not be registered on a wattmeter, and which may be omitted from the ordinary calculations of power.

EXAMPLE II.—*In the repulsion motor discussed on pp.* 156–160, *find the power communicated to the rotor at full load.*

The rotor current is represented in the stator by C_s of value 624 amps.

The rotor E.M.F. is represented in the stator by (see p. 153)

$$\mathrm{C}_s k^2 \cos^2 \lambda \mathrm{Z}_2 + j\mathrm{C}_1\mathrm{Z}_m \frac{n}{\sim} \cot \lambda$$

The required power is the scalar part of the vector product

$$\mathrm{P} = \mathrm{C}_s\left(\mathrm{C}_s k^2 \cos^2 \lambda \mathrm{Z}_2 + j\mathrm{C}_1\mathrm{Z}_m \frac{n}{\sim} \cot \lambda\right)$$

In order to get this we must have the relationship of all the vectors with respect to one axis of reference. As C_s occurs most often, we shall choose this and mark it c_s'.

Now, we have already found the relationship between C_s and C_1. It is

$$\mathrm{C}_s = (0{\cdot}96 - j0{\cdot}4)\mathrm{C}_1 \text{ (see p. 157)}$$

So

$$\mathrm{C}_1 = \frac{\mathrm{C}_s}{0{\cdot}96 - j0{\cdot}4}$$

Rationalizing we get

$$\mathrm{C}_1 = \frac{\mathrm{C}_s(0{\cdot}96 + j0{\cdot}4)}{1{\cdot}08} = \mathrm{C}_s(0{\cdot}89 + j0{\cdot}37)$$

or

$$\mathrm{C}_1 = 0{\cdot}89c_s' + 0{\cdot}37c_s''$$

From this knowledge and the fact that

$$\begin{aligned} Z_2 &= r_2 - jx_2 = 0{\cdot}003 - j\,0{\cdot}02 \\ Z_m &= r_m - jx_m = 0{\cdot}005 - j\,0{\cdot}073 \\ k^2 \cos^2 \lambda &= 1{\cdot}44 \\ \frac{n}{\sim} \cot \lambda &= 3 \end{aligned}$$

we get

$$\begin{aligned} \mathrm{P} &= c_s' \times c_s'(0{\cdot}0043 - j\,0{\cdot}0288) + c_s' \times (0{\cdot}89c_s'' - 0{\cdot}37c_s')(0{\cdot}015 - j0{\cdot}22) \\ &= (c_s' \times 0{\cdot}0043c_s') - (c_s' \times 0{\cdot}0288c_s'') + (c_s' \times 0{\cdot}19c_s') + (c_s' \times 0{\cdot}094c_s'') \end{aligned}$$

The first and third terms constitute the scalar part of this

$$\mathrm{P} = 0{\cdot}0043c_s^2 + 0{\cdot}19c_s^2$$

Now, c_s has been shown to be 624 amps., hence

$$\begin{aligned} \mathrm{P} &= 39 \times 10^4 \times 0{\cdot}0043 + 39 \times 10^4 \times 0{\cdot}19 \\ &= 1677 + 74{,}100 \\ &= 75{,}777 \text{ watts, of which 1677 watts are lost in rotor by resistance} \end{aligned}$$

As a check on this result it was also shown (p. 157) that C_s leads C_1 by 22°, and that its real value is 624 amps. And the E.M.F. balancing the rotor E.M.F.

$$\begin{aligned} = \mathrm{E}_1 &= \mathrm{C}_s k^2 \cos^2 \lambda \mathrm{Z}_2 + j\mathrm{C}_1 \mathrm{Z}_m \frac{n}{\sim} \\ &= c_s'(0{\cdot}0043 - j0{\cdot}0288) + (0{\cdot}094c_s'' + 0{\cdot}19c_s') \\ &= (2{\cdot}7' - 18'') + (58'' + 118{\cdot}6') \\ \mathrm{E}_1 &= 121{\cdot}3' + 40'' \\ e_1 &= 128 \text{ volts} \end{aligned}$$

and E_1 leads C_s by $\tan^{-1} \dfrac{40}{121{\cdot}3}$ (about 18°).

So $\mathrm{P} = e_1 c_s \cos 18° = 128 \times 624 \times 0{\cdot}95 = 75{,}800$ watts (nearly).

CHAPTER IX

LOCUS DIAGRAMS

Geometrical Constructions. — The ordinary vector diagram represents a number of quantities for certain definite conditions, so that all the quantities in the diagram are shown as having certain fixed values and definite relative phases.

If one of the quantities shown on such a diagram is altered in value, it will generally produce a change in some of the others, so that a new diagram would have to be drawn in order to represent the new set of conditions.

In most cases it is possible to show the loci followed by the various vectors, as one of them changes within certain limits, and by so doing to make it easy to obtain from a single diagram the changes undergone by the various vectors.

A locus diagram is thus founded on the ordinary vector diagram, but forms an extension of it, since it enables corresponding values of the vectors to be rapidly obtained for varying conditions of load or voltage, etc.

In the present chapter we shall consider the various cases which can be represented by the circle diagram; these will be found to include almost all classes of machines having either a constant current or a constant voltage.

When examined in detail, it is found that all alternating-current circle diagrams depend on a very few simple geometrical and electrical elements. These elements we have tried to arrange systematically under headings which give the conditions represented, and by a uniform system of lettering we have further endeavoured to make the diagrams to a large extent self-explanatory and easy of reference.

Each diagram is based on one of the two fundamental principles of an alternating-current circuit, viz.:

(*a*) The total applied voltage of a circuit may be considered as equivalent to two mutually perpendicular components, one in phase with, and one perpendicular in phase to, the current; and

(*b*) The current in a circuit may be resolved into two components, one component being in phase with the applied voltage, and one at right angles in phase to this voltage.

These relations may be expressed briefly in the notation employed in previous chapters. Symbolically

$$\mathrm{E} = \mathrm{C}(r - jx) \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$\mathrm{C} = \mathrm{E}(g + jb) \quad . \quad . \quad . \quad . \quad . \quad (b)$$

where E = voltage applied to the circuit
C = current flowing in the circuit
r = resistance
x = reactance
g = conductance $= \dfrac{r}{r^2 + x^2}$ (see p. 60)
b = susceptance $= \dfrac{x}{r^2 + x^2}$

Numerically, the relations (*a*) and (*b*) are written

$$e = c\sqrt{(r^2 + x^2)} \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$c = e\sqrt{(g^2 + b^2)} \quad . \quad . \quad . \quad . \quad . \quad (b)$$

The numerical form of these relations shows at once that the right-hand side represents components fulfilling the circle law, as will be more clearly seen from what follows.

It is necessary to call attention to the fact that the energy component of the voltage of a constant-voltage circuit is indicated in the following diagrams as the product of the current and a resistance (cr or $c\mathrm{R}$). This component voltage will, however, often be partly due to the introduction of an induced E.M.F. into the circuit (*e.g.* the induced E.M.F. due to rotation of a motor armature). It is consequently not necessarily equal to the product of the current and resistance of the circuit. In general, therefore, CR is to be taken as expressing the component of the voltage which is in phase with the current of the circuit, *i.e.* the total energy voltage.

The meaning of R in the above expression is not the actual value of the resistance of the circuit, but represents the value of a resistance of such magnitude that when multiplied by the current in the circuit, the product gives the value of the energy voltage. This resistance is sometimes called the *effective resistance,* and has been referred to at greater length on p. 56.

A similar use is made of the symbols Eg in the constant-current circuit. This is to be taken as representing the total energy current flowing, *i.e.* the component of the current which is in phase with the applied voltage.

The following conventions have been employed in drawing the diagrams which follow in this chapter:—

Capital letters (X, R) denote constant quantities (reactance, resistance). Small letters (x, r) denote the corresponding variable elements of a circuit.

The same meaning may be attached to the large and small letters indicating the current and voltage of the circuit, although the distinction follows from other considerations. Thus, in a constant-voltage circuit, the voltage will be indicated by a capital E. The sides of the voltage diagram are to be represented by the product of the *numerical value* of the current by an impedance, reactance, or resistance. The numerical value of a vector being always indicated by a small letter, it follows that a *small* letter will in this case always be employed for the current (which is variable) in the constant-voltage voltage diagram. From similar reasoning, it will be seen that a small e will indicate the voltage (which is variable) in a constant-current current diagram, in which the current will be indicated by a capital letter.

The letters O, A, B, etc., at the ends of the vector on the circle diagrams correspond to the letters at the ends of the portions of the circuit to which these vectors apply, so that the interpretation of any line on the circle diagram is at once obtained by a reference to the adjacent circuit diagram.

The arrow indicating current in the circuit diagram denotes the path, but not the direction of the current.

General Case of a Simple Circuit with Constant Applied Voltage.—Let Fig. 69 be the vector diagram of a circuit in which the applied voltage is represented by OB, and the current by OC. Resolving the voltage into components parallel and normal to the current, the numerical values of these components are *cr* and *cx*, if *r* and *x* are the "effective" resistance and reactance of the circuit. It is evident that the voltage will be capable of resolution into two mutually perpendicular components in this way, no matter what the actual values of *r* and *x* may be.

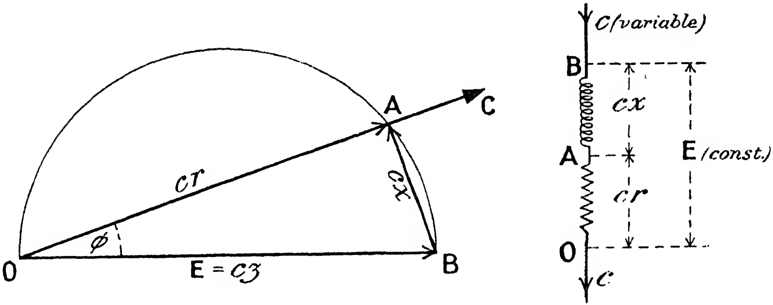

Fig. 69.—General diagram for a constant-voltage circuit.

Since the apex of a right-angled triangle will lie on the semicircle described on the hypotenuse of the triangle as diameter, it follows that for all values of *r* and *x* the point A in Fig. 69 will lie on the semicircle described on OB. Thus the semicircle is the locus of the point A when OA = *cr* and AB = *cx*, for the conditions of constant applied volts OB and variable values of *r*, *x*, and *c*.

For all conditions, the angle AOB is the angle of phase-difference between current and applied voltage.

The same result may be arrived at by employing the symbolic form of the voltage. The constant applied voltage E may be written

$$\begin{aligned} E &= c'(r - jx) \\ &= cr' - cx'' \end{aligned}$$

the meaning of which is that E is the vector sum of a voltage *cr*, and of a voltage *cx* perpendicular to *cr*, where *cr* is the component of E in phase with the current and $-cx''$ is the component perpendicular to the current. The sum of these components is constant, and when the component vectors are drawn in a diagram, they must form two sides of a right-angled triangle, having the constant voltage E as its hypotenuse.

In most of the problems occurring in practice, it naturally happens that either *r* or *x* remains constant, while the other function varies. It is then of use to determine the variations in magnitude and phase of the current produced by changes in the variable factor of the circuit.

The two cases where the resistance and reactance of the circuit are respectively constant may be considered separately.

Case 1.—**Applied Volts Constant. Resistance of Circuit Constant.**—In this case the reactance *x* is the variable element in the circuit. In the triangle OAB representing the total and component voltages, the side AB is drawn equal to *cx*, and will vary with either *c* or *x*. The side OA representing *c*R will vary only with the current, since R is constant. *Hence the line* OA *in Fig.* 70 *is proportional to the current in the circuit, and may be employed to show the current values.* For this purpose it will be convenient to alter the scale originally chosen for amperes, so that OA may represent the current directly in amperes, or else to alter the scale of the diagram so that OA may become equal to the current on the original scale of amperes.

The angle of phase difference between current and voltage is the angle AOB, marked ϕ on the diagram. The line OA may consequently be taken to represent the current in both magnitude and phase relation to the applied voltage. We may enumerate one or two further characteristics shown on the diagram.

Since the resistance is constant, the power in the circuit ($= c^2R$) will be proportional to c^2, *i.e.* to $(AO)^2$, or to OD, since this is proportional to $ec \cos \phi$.

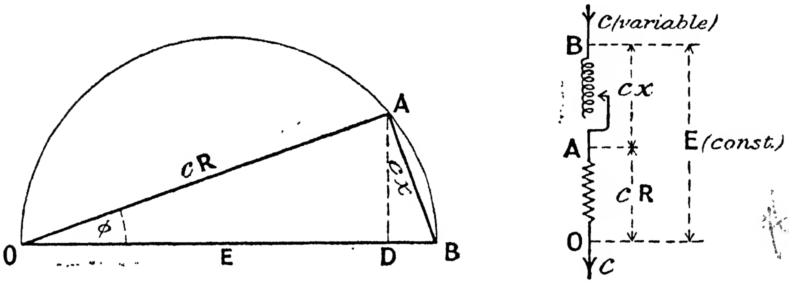

Fig. 70.—Constant-voltage circuit (Case 1).

The power-factor, $\cos \phi$, is proportional to OA.

The impedance of the circuit is proportional to $\dfrac{\text{voltage}}{\text{current}}$, *i.e.* to $\dfrac{1}{OA}$.

The variation of these quantities with the reactance or current may thus be easily traced.

Reactance Negative.—If the circuit possesses capacity, instead of inductance, the value of the reactance is

$$\frac{-1}{2\pi \sim K}$$

where K = capacity of the circuit in farads.

In this case x is therefore negative, and the reactance voltage $-xc''$ is a voltage xc rotated in a *positive* or counter-clockwise direction relative to the current vector. The voltage diagram for the circuit becomes in consequence similar to Fig. 71, if we take the applied voltage horizontal, as in the previous case.

This may be stated more briefly thus: If x is due to a capacity in the circuit, the symbolic impedance of the circuit is $Z = r - jx$, where

$$x = \frac{-1}{2\pi \sim K}$$

and the voltage applied $= E = c'(r - jx)$. The term $(-jc'x)$ is a positive quantity in the above expressions, and is measured upwards from A.

valid?

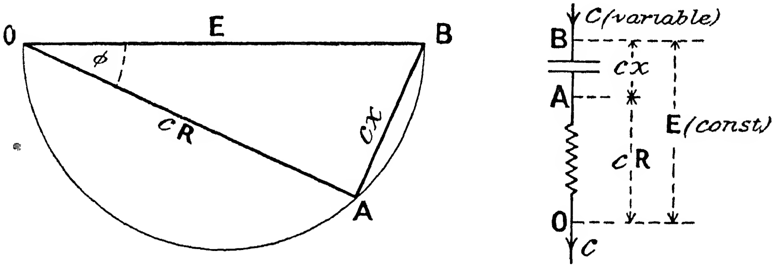

FIG. 71.—Constant-voltage circuit (Case 1), negative reactance.

In general, the value of the reactance is

$$x = 2\pi \sim L - \frac{1}{2\pi \sim K}$$

and the sign of this expression will determine whether the angle ϕ is to be measured above or below the horizontal voltage vector.

Case 2.—**Applied Volts Constant. Reactance of Circuit Constant.**—In this case r is the variable component of the impedance, and since X is constant, the current will be proportional to CX, *i.e.* to the vector BA (Fig. 72) in magnitude. Thus, BA *may be taken to represent the current to scale.*

It follows that when OB is taken as the constant applied voltage of the circuit (see Fig. 72), the current is represented *in magnitude* by the line BA and *in phase* by OA relative to the vector of applied volts.

By drawing the dotted line BV vertically from B and without any other change in the diagram, assuming BV to be the phase of the applied voltage, we can take the line AB as representing the current both in magnitude and in phase relative to the voltage BV.

With BV giving the phase of the applied volts (OB is, of course, equal to these volts in magnitude as before), we can summarize the lines showing the chief variable quantities in the circuit as follows:—

BA represents the current in magnitude and phase.

AD = BA cos ϕ is proportional to the power in the circuit.

A′D′ shows maximum output.

Angle ABV = angle of phase difference between current and voltage.

Power-factor of circuit is proportional to OA.

Impedance of circuit is proportional to $\frac{1}{\mathrm{BA}}$.

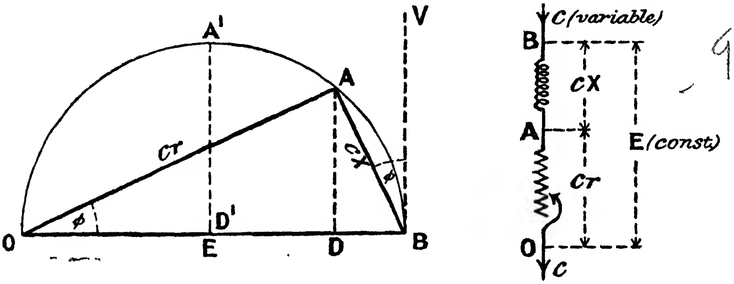

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FIG. 72.—Constant-voltage circuit (Case 2).

From the two diagrams already given (Figs. 70 and 72) it is evident that in constant-voltage circuits with constant resistance, horizontal projections of the current line (*e.g.* OD) represent watts. In circuits with constant reactance, vertical projections (*e.g.* AD) give the watts. These rules may be applied in the subsequent diagrams, although it is not specially pointed out in each case.

IMPEDANCES IN SERIES IN CONSTANT-VOLTAGE CIRCUIT.

Resistance in Series with Impedance (Constant Applied Voltage).—The total voltage applied to the circuit being assumed constant, the voltage triangle for the whole circuit must lie in the semicircle described on a line representing the constant voltage, for all possible values of the component resistances and reactances.

Let the circuit be composed of an impedance (resistance r and reactance x), and of a constant resistance R_1 in series with the impedance.

Then, if a triangle OAB be constructed as in the foregoing diagrams (Figs. 70 to 72), the line OA would represent the total energy voltage $(r + R_1)c$, while BA would give the idle voltage xc.

Special interest attaches only to those cases where (as generally happens) the impedance has one of its components r or x constant. These cases are considered separately.

Case 3. — **Applied Voltage Constant. Two Constant Resistances in Series with Variable Reactance.**—This is a development of Case 1. The resistance R_1 will always be a definite fraction of the total resistance $R + R_1$ in the circuit, and will absorb a constant fraction of the energy voltage of the circuit. The following construction enables the distribution of the component voltages of the circuit to be traced as x varies.

Draw a semicircle on the line OB representing the constant applied voltage (see Fig. 73). Let OA represent the total voltage overcoming resistance = $c(R_1 + R)$. Divide OA at F, in the ratio of R_1 to R, so that

$$\frac{OF}{FA} = \frac{R_1}{R}$$

Draw FD perpendicular to OA to cut OB in D. On OD describe the semicircle OFD. This semicircle will divide the vector OA in such a way that OF always gives the voltage cR_1, and FA the voltage cR. This is evident from the similarity of the triangles OFD, OAB, from which the ratio

$$\frac{\mathrm{OF}}{\mathrm{OA}}$$

is seen to be constant for all positions of A.

The voltage E_1 across the impedance $(R - jx)$ is FB, and the current is proportional to the length of OA. The angle of lag of the whole circuit is that marked ϕ, while that of the portion of the circuit containing the impedance $R - jx$ alone is ϕ'.

OH is proportional to the watts expended in the resistance R_1.

OK is proportional to the watts in the whole circuit.

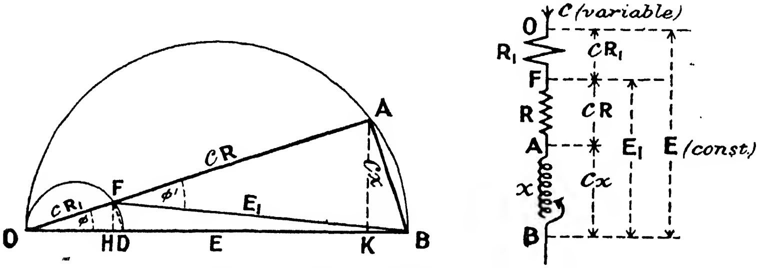

FIG. 73.—Constant-voltage circuit (Case 3).

This construction is often of value where a circuit is supplied through a line having a resistance represented by R_1, or when the resistance of the winding of some apparatus is to be taken into account.

An alternative construction to the one shown in Fig. 73 is sometimes adopted for separating the voltage lost in a part of the circuit having constant resistance. Instead of setting off OD on the left of the diagram, and describing a semicircle on OD, we may mark off the same length (BD in Fig. 74 = OD in Fig. 73) from B and draw a semicircle on OD. The length FA is seen from the similar triangles OAB, OFD, to represent the voltage spent in the resistance R_1. This construction is less satisfactory than the one previously given, inasmuch as it does not enable us to obtain directly the voltage of the remaining part of the circuit (viz. the voltage $c(\mathrm{R} - jx)$) as a single line on the diagram.

Figs. 73 and 74 will be seen to be really identical, except that $c\mathrm{R}$, instead of $c\mathrm{R}_1$, is measured off as the diameter of the semicircle OD. OH now represents the watts in the impedance $\mathrm{R} - jx$, while OK again represents the total power in the circuit.

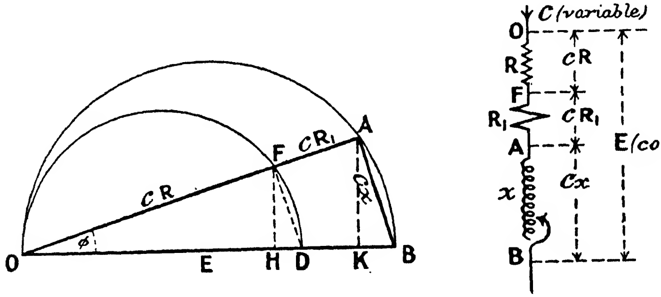

FIG. 74.—Constant-voltage circuit (Case 3), alternative construction.

Case 4.—**Applied Voltage Constant. Constant Resistance in Series with Constant Reactance and Variable Resistance.**—This case differs from the previous one, because the current is now proportional to the reactance voltage BA instead of to AO. The voltage absorbed in the resistance R_1 is obtained by drawing a semicircle OFD on a diameter perpendicular to OB. In order to obtain this semicircle, set up OD (see Fig. 75) perpendicular to OB, and mark off the length of OD equal to the fraction $\frac{\mathrm{R}_1}{\mathrm{X}}$ of OB. The length OF cut off from OA by the semicircle described on OD will always give the voltage spent in the resistance R_1, since with the ratio of diameters of the semicircles OB, OD, just given, it is evident that OF will represent $c\mathrm{R}_1$ volts to the same scale that AB represents $c\mathrm{X}$ volts. As before, FB represents the voltage E_1 across the impedance $z = r - jX$ for any value of the current in the circuit.

The reason for this construction follows from the similarity of the triangles DOF and OBA, from which it is seen that OF is proportional to AB. But AB represents the current in the circuit to some scale, since it is equal to the reactance voltage in a circuit of constant reactance. Hence OF is proportional to the current. Also, from the construction adopted, its length represents the voltage CR_1 to the same scale as that to which AB represents the voltage cX.

As, therefore, the angle marked ϕ in Fig. 75 is the angle of lag for the whole circuit, and ϕ_1 is the angle of lag in the impedance $r - jX$, E_1 is the voltage spent in this impedance.

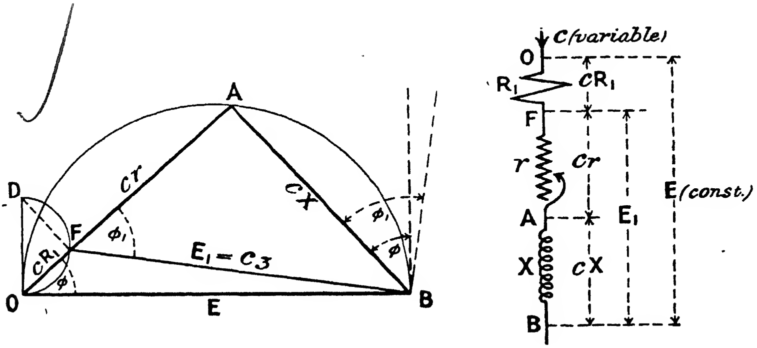

FIG. 75.—Constant-voltage circuit (Case 4).

The watts spent in the resistance R_1 are proportional to the height of F above OB, the watts in the whole circuit being represented (to the same scale) by the height of A.

If the drop in the constant resistance R_1 in the case just given (Fig. 75) is measured off along AO from A, instead of from O, so that AF in Fig. 76 represents this voltage, the locus of F is again the arc of a circle (not in this case a semicircle). This circle passes through O and B, and has its entre a distance $\text{OB} \times \dfrac{R_1}{2X} = \tfrac{1}{2}\text{OD}$ below, but on the same vertical centre-line as, the semicircle OAB.

AF now represents the drop in the constant resistance R_1, while OF is the variable energy voltage in the circuit.

The proof of this construction is as follows. The sides of the triangle BFA will all be proportional to the current in the circuit, since each represents a voltage equal to the product of the current and a constant resistance or reactance, or impedance. Hence the triangle must remain similar in shape for all values of the current. The angle BFA remains, therefore, constant, and its supplementary angle BFO must be also constant, and must therefore move in the arc of a circle passing through B and O.

The height of F above OB will represent the watts in r,

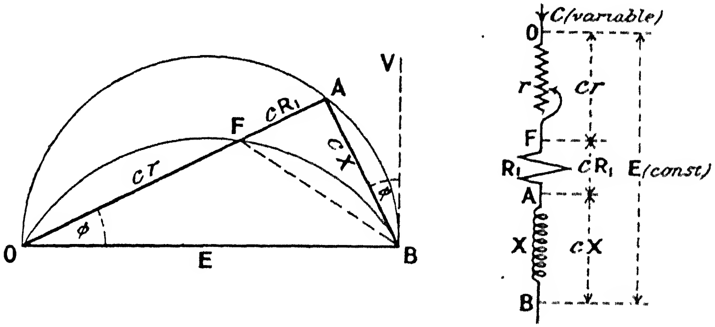

FIG. 76.—Constant-voltage circuit (Case 4), alternative construction.

and the difference between the ordinates of F and A the watts spent in R_1

Often cases occur in which there are two constant resistances R_1 and R_2 in the circuit, as well as a variable energy voltage (*e.g.* the field and armature resistances of a series motor). In this case, we can show the voltage lost in each resistance by drawing two semicircles (see Fig. 77), one having a diameter OD′ such that the semicircle cuts off a length proportional to the voltage lost in both resistances together and the other semicircle on OD, showing that lost in one of them. The intercept between the semicircles (FG in Fig. 77) then gives the drop in the second resistance, while OF gives the drop in the first.

The conditions represented by a circuit with constant reactance and variable resistance are those met with in the case of motors, transformers, etc., where the inductance of the circuit may be taken as constant and the variable voltage *cr* includes the energy voltage of the motor or the voltage which produces the secondary voltage of the transformer.

A very good instance of a circuit illustrated by Case 4, Fig. 75, is that of an alternator with constant excitation and variable non-inductive load. The resistance and reactance of the alternator armature are very nearly constant, and are connected in series with a variable non-inductive resistance, indicated by r in the diagram. The changes of terminal volts and current for varying values of the load resistance can thus be easily traced from the diagram.

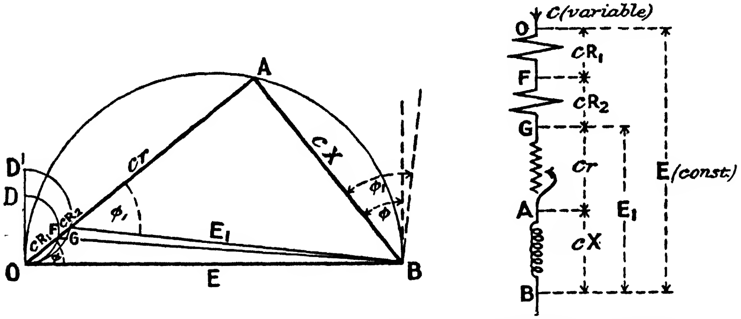

FIG. 77.—Constant-voltage circuit (Case 4), with additional resistance.

Another application may be made of the diagram given in Fig. 76. It may be made to represent the conditions of a circuit of constant power-factor in series with a constant resistance. In this case the triangle FAB is taken to show the voltages in the circuit in which the ratio of r to x is constant, so that the ratio of FA to AB is constant for any values of current and for any values of r or x.

The volts spent in the constant resistance in series with this circuit are shown by OF, the length of which will also represent the current both in magnitude and phase. The total applied volts of the system are constant and given by OB.

Reactance in Series with Impedance in Constant Voltage Circuit.—The construction for showing the variation in current and distribution of voltage in the circuit is very similar in this case to that just considered, where a resistance was in series with an impedance.

Case 5.—**Applied Voltage Constant. Constant Reactance in Series with Constant Reactance and Variable Resistance.**—In this case the total reactance of the circuit is constant, and the reactance voltage line AB is to be divided into a constant ratio $\frac{X}{X_1}$, since its whole length is proportional to the current, and both X and X_1 are constant.

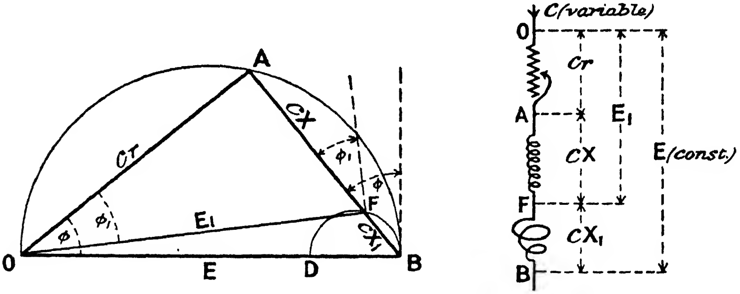

FIG. 78.—Constant-voltage circuit (Case 5).

The construction is thus exactly analogous to Case 3, the difference being that in the present instance the voltage spent in the constant reactance X_1 is marked off on the reactance voltage line BA (see Fig. 78), whereas in the former case the voltage spent in the resistance R_1 was marked off from the resistance voltage line OA.

The voltage E_1 across the impedance $r - jX$ is given by the line OF, and the voltage spent in the reactance X_1 in series with it by BF. The current in the circuit is proportional to BA, and the angle of lag of the whole circuit is that marked ϕ. The angle of lag of the circuit whose voltage is E_1 is that marked ϕ_1.

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As in Case 3, CX, instead of CX_1, may be marked off from B, thereby altering the ratio of the diameter of the semicircles of the diagram, but at the same time altering the signification of the line OF, which then represents the voltage across $r - jX_1$ instead of $r - jX$.

Case 6.—**Applied Voltage Constant. Constant Reactance in Series with Constant Resistance and Variable Reactance.**—In this case the current is proportional to OA (see Fig. 79), and the voltage spent in the constant reactance X_1 must be represented by a length BF cut off from BA, BF being proportional to the current, *i.e.* to OA. The construction for doing this is similar to that employed in Case 4, and is shown in Fig. 79, where BD is set up vertically at B of such length that

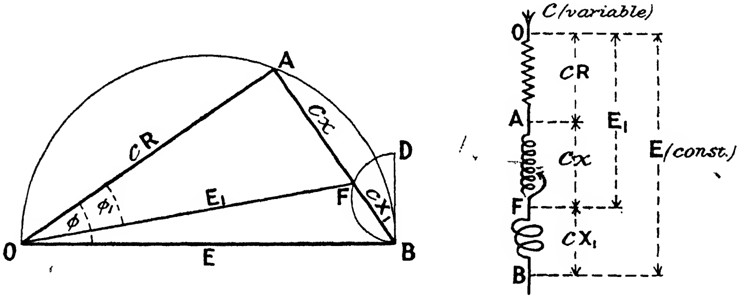

FIG. 79.—Constant-voltage circuit (Case 6).

$$\frac{BD}{OB} = \frac{X_1}{R}$$

BF represents the voltage spent in X_1, while OF is the voltage E_1 across the impedance $R - jx$. ϕ is the angle of lag in the whole circuit, so that the current is represented in phase and magnitude by OA.

ϕ_1 is the angle of lag in the portion of the circuit containing the impedance $R - jx$.

By measuring off the voltage CX_1 ($= FB$) from A instead of from B, we obtain a construction similar to that in Fig. 76, the point F then moving in the arc of a circle of which OB forms a chord.

The construction and significance of the figure will be evident from what has already been said in connection with Fig. 76.

It will also be seen that the same construction may be used to represent the conditions of a circuit of constant power-factor, but variable load, supplied in series with a constant reactance.

Negative Reactance in Series.—In the two diagrams just given, it was assumed that the reactance of the receiving circuit and the reactance in series with it were both of the same sign (in the diagrams both reactances are positive, *i.e.* both were due to inductance). If the constant reactance in series with the impedance is opposite in sign to that of the receiving circuit, the total reactance voltage of the circuit will be the *difference* between the two reactance voltages instead of their sum.

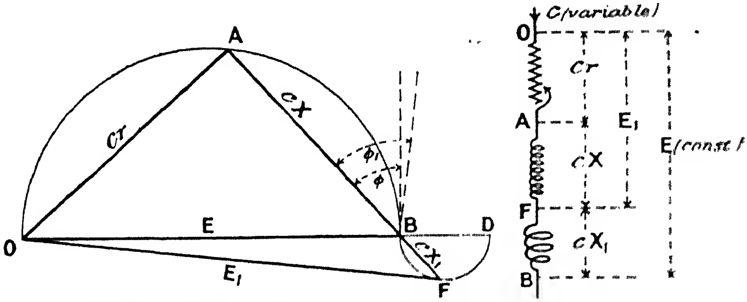

FIG. 80.—Constant-voltage circuit (Case 5), negative reactance.

If the sign of the reactance X_1 is negative, the construction corresponding to Fig. 78 is carried out according to the same rule as before, but the diameter of the semicircle which is to separate the reactance voltage CX_1 must be drawn in the reverse direction (indicating the negative sign of the reactance), so that the figure becomes as shown (Fig. 80).[1]

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[1] Diagrams 80 and 81 should have X_1 represented by a condenser in order to show a negative capacity. As the same diagrams are employed to show the construction for more usual cases (76 and 77) where X_1 is a positive reactance, it was thought better to draw them as shown.

It is here evident that the voltage E_1 across the portion of the circuit AF containing r and X alone is greater than the total voltage E given to the circuit including r, X, and X_1.

We have, in fact, a case of voltage resonance.

A similar change occurs in the diagram shown in Fig. 79 when X_1 becomes negative. As before, the semicircle indicating voltages in X_1 must be drawn on a diameter marked off in the opposite sense. With this change, the diagram becomes that shown in Fig. 81. Again the effect of resonance is seen in producing a greater voltage in a portion of the circuit than the total applied voltage.

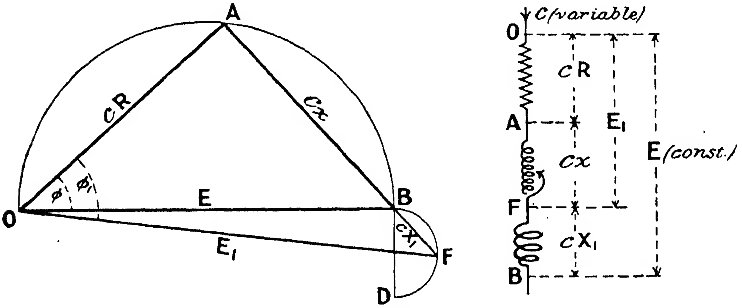


FIG. 81.—Constant-voltage circuit (Case 6), negative reactance.

The effect of the construction given in the last two figures (Figs. 80 and 81) is to subtract a negative reactance voltage from the circuit represented by the fundamental triangle OAB. The subtraction of a negative voltage is equivalent to the addition of a positive voltage. Accordingly, the Figs. 80 and 81 may also represent the addition of the voltage required to overcome a constant reactance due to self-induction (say magnetic leakage) to the constant voltage OB, the line OF representing in this case a variable terminal voltage required to maintain a constant voltage OB across the reactance $r - jX$, or $R - jx$ respectively. The two diagrams just given are accordingly those fulfilling the conditions of Cases 8 and 9, given later, showing a constant-voltage circuit supplied through a constant reactance.

Two Impedances in Series in Constant-Voltage Circuit.—If a circuit contains two impedances in series, the diagram showing the idle and energy components of the total voltage will be given by the usual construction of a triangle in a semicircle whose diameter represents the total applied voltage. Now, divide the energy voltage at a point F (Fig. 82), so as to make the ratio

$$\frac{AF}{FO}$$

equal to the ratio of the energy voltage of the first circuit to that of the second. Similarly, mark a point G in the idle voltage line such that

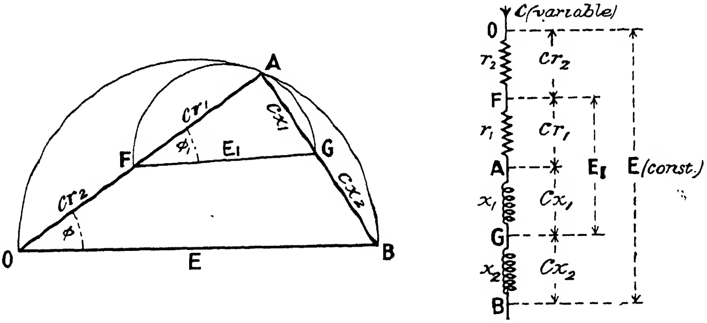


Fig. 82.—Constant-voltage circuit—two impedances in series.

$$\frac{AG}{GB}$$

is the ratio of the reactance voltages in the first and second impedances respectively. Joining F and G, we have the triangle FAG, representing by its sides the total and component voltages of the first impedance. Since this triangle contains a right angle, a semicircle may be described on its base to pass through A.

The useful developments of the diagram just given (Fig. 82) occur when only one of the four quantities r_1, r_2, x_1, x_2, is a variable and the others are constant.

Case 7.—**Applied Voltage Constant. Constant Impedance in Series with Constant Reactance and Variable Resistance.**—This is the most useful case, where the energy voltage in one part of the circuit varies (we will assume this to be represented by variation of r_2), as shown in Fig. 83.

Since the reactances of the circuit remain constant, we can show the locus for the point G, which divides the reactance drop in the two impedances by drawing a semicircle in the manner shown in Fig. 78, for a circuit with constant reactance. If now R_1 is the resistance in the impedance which remains constant, the length of AF, which represents the voltage overcoming this resistance, will be proportional to the current, *i.e.* to the length of BA. The point F which fulfils this condition will travel in a circle passing through O and B, and having its centre on the same vertical centre-line as the semicircle on OB, as in Fig. 76 (p. 196, *ante*). The line FG evidently gives the voltage of the constant impedance, and OF the variable energy voltage in the circuit. The construction is, in fact, seen to be a combination of those already given in Figs. 76 and 78.

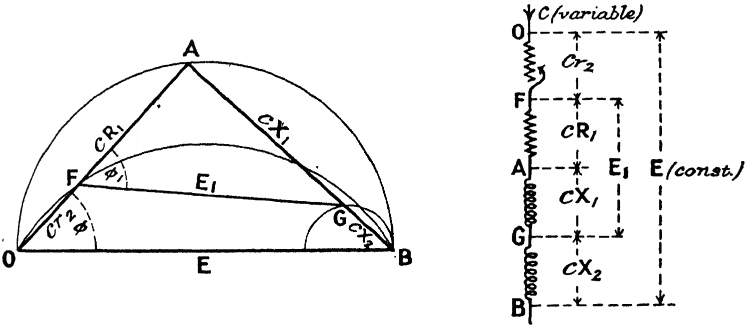


FIG. 83.—Constant-voltage circuit (Case 7).

By interchanging the positions of the vectors representing the voltage in the first and second impedance, so that the variable energy voltage cr_2 is measured from A, instead of from O, we obtain the result shown in Fig. 84, where the locus of F becomes a semicircle with vertical diameter.

We have, in this case, a combination of the conditions shown in Fig. 75 (p. 195, *ante*) and Fig. 78. FG now represents the voltage in the impedance which contains the variable element. The above are useful combinations of Cases 4 and 5, and often occur in practice, as, for instance, in the case of the series alternating-current motor, working on a constant voltage circuit. The symbols would then have the following significance:—

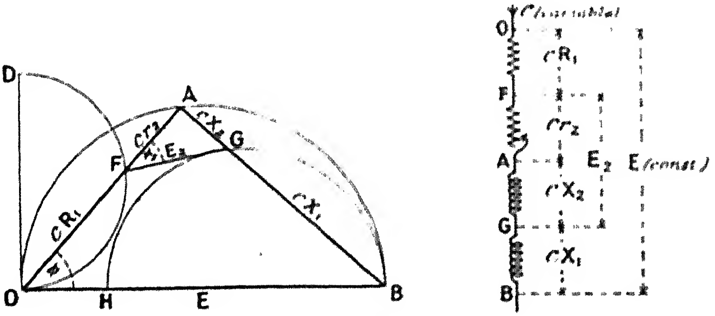


FIG. 84.—Constant-voltage circuit (Case 7), alternative construction.

R_1 = sum of armature and field resistances.
cr_2 = voltage induced in armature by rotation.
cX_2 = reactance voltage of field.
cX_1 = voltage due to armature self-induction.

It will be easy to construct the appropriate diagram to show the case of a variable reactance in a circuit containing a constant impedance and resistance. A special diagram is consequently not shown for this combination.

Reactance or Resistance in Series with Constant-Voltage Circuit.—In applying locus diagrams to electrical circuits it is sometimes of advantage to take the voltage across the impedance formed by the main portion of the circuit as constant, and to add on to this voltage the additional volts required to send the current through a constant resistance in series with the main impedance. A case of this kind would be a diagram showing the relations for a transformer to give a constant output voltage, the voltage lost in the windings being then added on to the constant voltage to give the primary terminal volts.

Two constructions of this kind have already been given in Figs. 80 and 81 for the addition of the voltage lost in a reactance in series with a constant voltage circuit, for the cases where the resistance and reactance respectively of the main impedance were constant.

Case 8.—**Constant Reactance in Series with Constant-Voltage Circuit. Resistance of Constant-Voltage Circuit Constant.**—The circle diagram for this case is that already given in Fig. 81.

Case 9.—**Constant Reactance in Series with Constant-Voltage Circuit. Reactance of Constant-Voltage Circuit Constant.**—The circle diagram for this case is Fig. 80.

It now remains to indicate the corresponding construction for representing the case where a resistance is put in series with a constant-voltage circuit.

Case 10.—**Constant Resistance in Series with Constant-Voltage Circuit. Resistance of Constant-Voltage Circuit Constant.**—The constant voltage is represented by OB (see Fig. 85), and the current is proportional to the line OA, since the resistance R is assumed constant. The voltage absorbed in the constant resistance R_1, in series with the circuit, must be proportional to OA, and also in phase with the voltage cR represented by it.

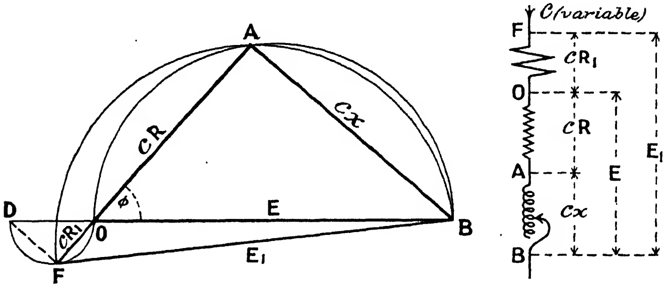


FIG. 85.—Resistance in series with constant-voltage circuit (Case 10).

Produce BO and mark off OD so that

$$OD = OB\,.\,\frac{R_1}{R}$$

and on OD describe a semicircle. It is easily seen that OF will be the line fulfilling the conditions just stated in order that it may give the voltage cR_1.

The total terminal voltage of the circuit is FB.

If the terminal voltage FB were maintained constant, the point A would move in semicircle described on FB, instead of that described on OB. The line FA would not then pass through O for all values of the current, so that the semicircle DFO would no longer show the voltage cR_1. In fact, O would then move on the circumference of a semicircle, having its diameter measured from F along FB, the conditions being those shown in Fig. 73, Case 3.

Case 11.—**Resistance in Series with Constant-Voltage Circuit. Reactance of Constant-Voltage Circuit Constant.**—In this case the current in the circuit is proportional to AB (see Fig. 86), so that the added voltage cR_1 must be proportional to AB. This result is obtained by drawing the semicircle OFD with a diameter perpendicular to OB so that

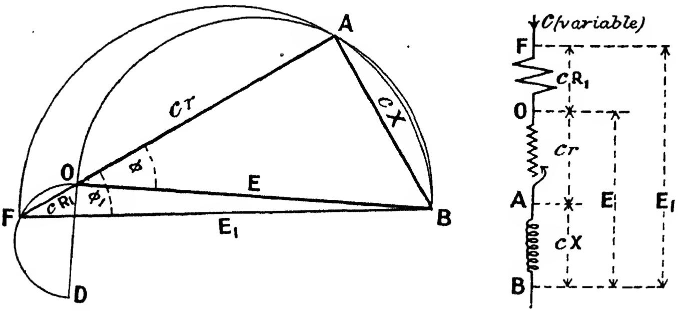


FIG. 86.—Resistance in series with constant-voltage circuit (Case 11).

$$OD = OB\,\frac{R_1}{X}$$

OF is then the resistance voltage cR_1 and the total terminal voltage is FB.

If the total applied voltage were maintained constant, A would move in a semicircle on FB instead of the semicircle BAO. The semicircle OFD would then no longer show variations of the voltage cR_1, but O would move on a semicircle instead of F, since the conditions would then be those of Case 4 (Fig. 75).

Case 12.—**Impedance in Series with Constant-Voltage Circuit. Resistance or Reactance of Constant-Voltage Circuit Constant.**—From the constructions already given, there will be no difficulty in making the diagram for this case. An addition has to be made to both the resistance voltage line AO, and also to the line of reactance voltage AF.

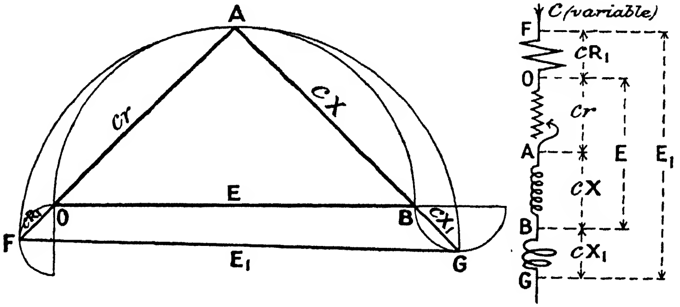


Fig. 87.—Impedance in series with constant-voltage circuit (Case 12).

As an illustration, Fig. 87 shows the addition of reactance and resistance in series with a circuit of constant reactance. If the circuit had had a constant resistance, the small semicircle on the right would have had a vertical diameter, and that on the left a horizontal diameter.

The terminal applied volts are given by FG, and a semicircle described on this line would give the locus of A if the terminal voltage were maintained constant.

Circuits in Parallel on Constant Voltage.—If we have two circuits connected in parallel to the same constant voltage supply, we could represent the conditions in the two circuits by drawing two semicircles, each having the same diameter and each showing the voltage relations for one of the circuits, as has been done in Fig. 88.

A diagram of this kind becomes useful where there is only one of the component voltages in each circuit which varies independently of the current.

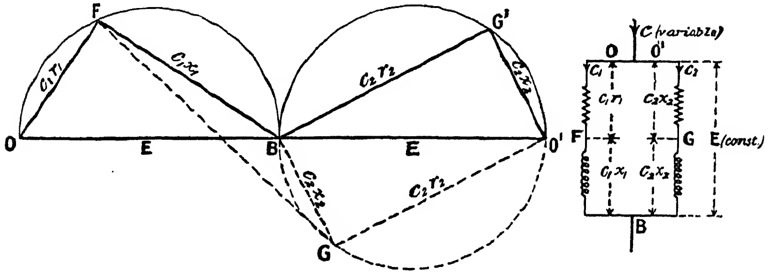


FIG. 88.—Circuits in parallel on constant voltage.

Case 13.—**Two Circuits in Parallel with Constant Applied Voltage. Reactance of both Circuits Constant.**—The most usual case will be where there is a variable energy voltage in the circuit. Let this be represented by considering r_2 to be variable.

By inverting the semicircle shown on the right in Fig. 88, the two voltages c_1x_1 and c_2x_2 are obtained adjacent and consequent.

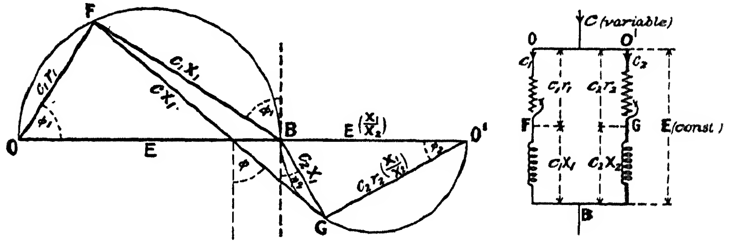


FIG. 89.—Constant-voltage circuits in parallel (Case 13).

Each of these voltages is proportional to the current in one of the parallel circuits. Each might therefore be taken to represent the current to scale. The scale would, however, be different in the two circuits, unless the reactances in both happened to have the same value. In order to obtain the resultant of the two currents, we must obtain vectors which shall represent the currents in both circuits to the *same scale.* This may conveniently be done by altering the scale of one of the circuit diagrams, say the diagram for circuit (2), so that the vectors c_1x_1 and c_2x_2 may both represent the current to the same scale. This change is accomplished by altering the diameter of the second semicircle, so that

$$\frac{\text{diameter of semicircle (1)}}{\text{diameter of semicircle (2)}} = \frac{\text{reactance of circuit (2)}}{\text{reactance of circuit (1)}}$$

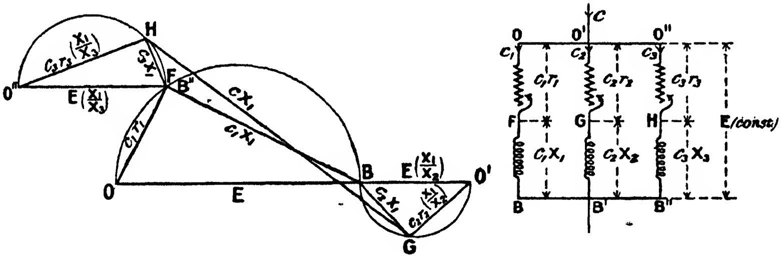


FIG. 90.—Three constant-voltage circuits in parallel (Case 13).

This change is shown as carried out in Fig. 89, where the lines FB and BG represent the currents in circuits (1) and (2) respectively. The resultant current in the combined circuit is represented by the line FG. It is to be noticed that all the lines in the triangle BGO′ have been reduced from their values in Fig. 88 in the ratio $\frac{X_1}{X_2}$. Thus the current line BG is

$$c_2X_2\left(\frac{X_1}{X_2}\right) = c_2X_1$$

as marked on the diagram.

If three circuits are in parallel, we can only extend the construction by adding further semicircles, in such a way that the lines representing current are drawn consecutively, so that they may be combined by a single line to show their resultant. The diameters of these circles will all represent the same voltage, but in order that the current scale may be the same in each case, the length of the diameters must be drawn in accordance with the rule explained in connection with the preceding diagram. The diagram then takes the form shown in Fig. 90.

Case 14.—**Two Circuits in Parallel with Constant Applied Voltage. Resistance of both Circuits Constant.**—The diagram for this case is exactly analogous to the preceding one. The only change is due to the fact that the lines OF, O′G, Fig. 89, will now be proportional to the currents in the two circuits. The second semicircle must consequently be still inverted but placed to the *right* of the first semicircle, so that the currents may be directly added. Also, we shall have the following rule for the relative diameters of the semicircle to allow of a common scale of currents to be applied to both :—

$$\frac{\text{diameter of semicircle (1)}}{\text{diameter of semicircle (2)}} = \frac{\text{resistance of circuit (2)}}{\text{resistance of circuit (1)}}$$

The construction and use of the diagram will be clear from the previous case.

Case 15.—**Two Circuits in Parallel with Constant Applied Voltage. Resistance of One Circuit and Reactance of other Circuit Constant.**—The reactance voltage of one circuit and the energy voltage of the other are now proportional to the respective currents. In order to combine the currents, the semicircle drawn inverted in Case 13 must be turned through a right angle, so as to come under the other as shown in Fig. 91. The currents may now be directly combined and represented by the line FG if the voltage scales of the diagrams are chosen so that

$$\frac{\text{diameter of semicircle (1)}}{\text{diameter of semicircle (2)}} = \frac{\text{resistance of circuit (2)}}{\text{reactance of circuit (1)}}$$

Simple Circuits with Constant Currents.—In the case of the constant-voltage circuit, we resolved the voltage into two components respectively in phase, and at right angles to the current, constructing a right-angle triangle in a semicircle to represent these voltages. The expression for the relation between these voltages, given in symbolic form, is

$$E = C(r - jx)$$

In a similar way, in the case of the constant-current circuit, we shall resolve the total current into energy and idle components respectively in phase and at right angles in phase to the applied voltage of the circuit. We thus again obtain a right-angle triangle on a fixed base (see Fig. 92). The triangle of resolved currents will again lie within a semicircle, and the relation between them and the constant total current may be expressed in the usual symbolic form

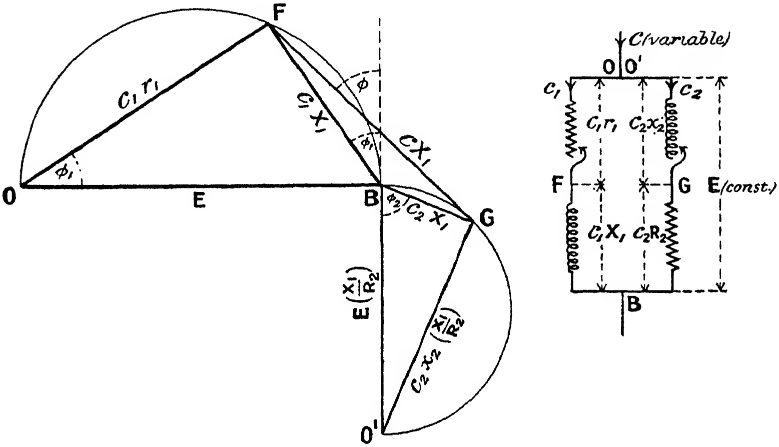


FIG. 91.—Constant-voltage circuits in parallel (Case 15).

$$C = E(g + jb)$$

where $g = \dfrac{r}{r^2 + x^2}$ is the conductance of the circuit

and $b = \dfrac{x}{r^2 + x^2}$ is its susceptance

Let OB in Fig. 92 represent the constant current in the circuit, and OE the applied voltage. Resolving OB into its components OA, AB, which are the energy and idle currents of the circuit, the point A will always lie on the semicircle

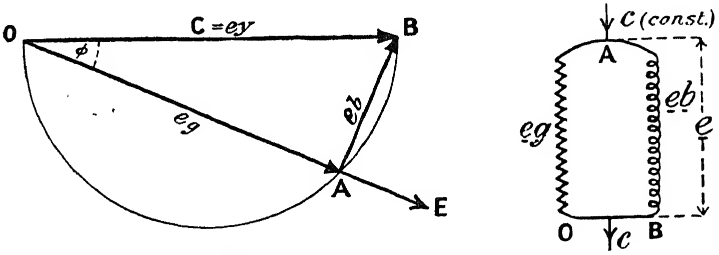


FIG. 92.—Constant-current circuit.

described on OB. The applied voltage is in this case variable.

Corresponding to the two special cases in the constant-voltage diagram, we may consider separately the current diagram when (1) the conductance is constant; (2) the susceptance is constant. In either of these two cases the side of the triangle which contains the constant function may be made to represent the value of the applied voltage, since it will represent this voltage, multiplied by a constant quantity.

Case 16.—**Current Constant. Conductance of Circuit Constant.**—The diagram for this case is shown in Fig. 93. The

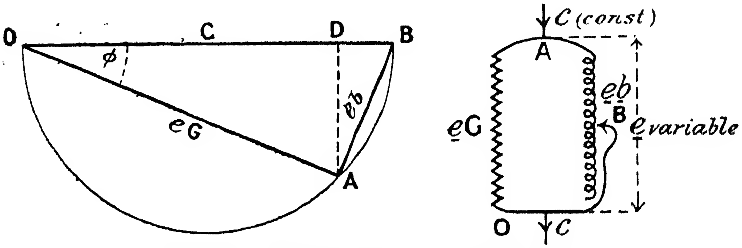


FIG. 93.—Constant-current circuit (Case 16).

applied voltage of the circuit is proportional to the line OA, and is also represented by OA in relative phase when the total current is given by OB. The scale of volts may accordingly be so chosen that OA becomes equal to the circuit voltage. We thus have in the diagram, Fig. 93—

Applied volts given by OA in phase and magnitude.

Power given by OA cos ϕ = OD, which is proportional to $(OA)^2$.

Power-factor = cos ϕ, which is proportional to OA.

Impedance of circuit proportional to OA.

Case 17.—**Current Constant. Susceptance of Circuit Constant.**—The applied voltage is now represented by the side BA in magnitude, and also in phase relative to the current vector if the latter is drawn vertically as shown in Fig. 94 by the dotted line BC. We now have—

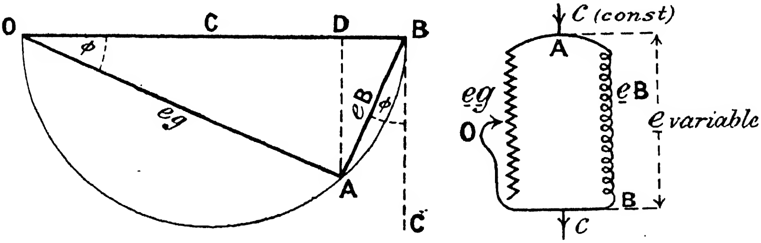


Fig. 94.—Constant-current circuit (Case 17).

Applied voltage proportional to AB.

Power proportional to AB cos ϕ, *i.e.* to AD.

Angle of lag ABC.

Case of Circuit with Capacity.—In a circuit having capacity, the idle current, which in this case is the charging current $2\pi \sim Ke$, will be in advance of the applied voltage in phase; in other words, the susceptance is negative. The current triangle with its enclosing semicircle will therefore be drawn above the vector of total current, instead of below.

The relations between the lines in the figure and the quantities represented by them in the circuit are exactly as already discussed. This case is evidently analogous to the example of negative reactance in the constant-voltage circuit given in Fig. 71.

Constant-Current Circuits. Branch Circuits in Parallel.—The development of the constant-current diagram for circuits in parallel on the same lines as the constant-voltage diagram for circuits in series should present no difficulty. It need only be remembered that in the constant-current diagram impedances in series will correspond to impedances in parallel in the constant-voltage circuit. Similarly, branch circuits in parallel forming a constant-current circuit will replace circuits in series in the constant-voltage diagram. The series of constant-voltage diagrams already given may thus be translated into constant-current diagrams. In each case a reactance voltage will become a susceptance current, and a resistance voltage will be replaced by a conductance current. Examples of this use of the diagrams will be found in Chapter X.

Case 18.—**Circuit having Constant R and L supplied from Source having Variable Frequency.**—This is a special case of the application of the vector diagram to a circuit having variable frequency. The construction is also employed in showing the relation between the current and slip of an induction motor.

Taking as the more typical instance an alternator[1] having constant excitation, and connected to a circuit of constant resistance and coefficient of self-induction, the alternator induced voltage E will be directly proportional to the speed.

Let x, R be the total reactance and resistance of the load circuit (including the alternator armature), and C the current produced by the voltage E.

$$\mathrm{E} = \mathrm{C}(\mathrm{R} - jx) \text{ or numerically } e = c\sqrt{\mathrm{R}^2 + x^2}$$

since both E and x are directly proportional to the speed, the numerical value of the current will be proportional to the expression shown below.

$$c \propto \frac{x}{\sqrt{\mathrm{R}^2 + x^2}}$$

x being proportional to E for any given excitation within the limits of saturation.

---

[1] See an article in the *Electrical Engineer*, London, August 12, 1904.

Draw OD, OB at right angles (see Fig. 95), and make OD equal to x ($=$ reactance at full speed), OB equal to R.

As the speed falls, the reactance x will assume a smaller value represented by OD′.

$$\text{Joining D}'\text{B, the line D}'\text{B} = \sqrt{\text{R}^2 + x^2}$$

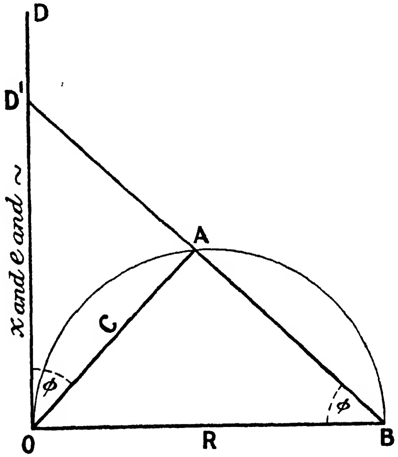


Fig. 95.—Variable frequency circuit (Case 18).

The current in the circuit has been shown to be proportional to

$$\frac{x}{\sqrt{\text{R}^2 + x^2}} \textit{ i.e.} \text{ to } \frac{\text{OD}'}{\text{D}'\text{B}} \text{ or } \sin \phi$$

Now, a line drawn from O perpendicular to BD′ will mark off a triangle OAB similar to the triangle D′OB. Also, as D′ changes in position, the triangle OAB will always be a right-angle triangle described on the constant hypotenuse OB, so that the locus of A will be the semicircle described on OB.

Also $\sin \phi = \dfrac{\text{OA}}{\text{OB}}$ or, since OB is constant, we have

$$c \propto \sin \phi \propto \text{OA}$$

whence OA can be taken to represent the current in the circuit in magnitude. It will also represent it in phase relative to the voltage (shown by OD′), since the angle AOD′ = angle D′BO = ϕ.

That ϕ is the angle of lag in the circuit is seen from the fact that

$$\tan DBO = \frac{r}{R}$$

As already stated, the height of OD′ is proportional to the total generated volts, and may thus be taken as representing the voltage in the circuit.

It may be interesting in conclusion to mention a few typical cases where the diagrams given above find a direct application.

Fig. 76.—Correction for the copper losses in Heyland's diagram for the induction motor.

Fig. 77.—Alternator load characteristic with constant excitation.

Figs. 78 and 84.—Series alternating-current motor.

Fig. 89.—Single-phase induction motor considered as having oppositely rotating stator fields.

Evidently many cases of drop in transmission lines will be directly represented by such figures as Nos. 73, 75, 78, 87, etc.

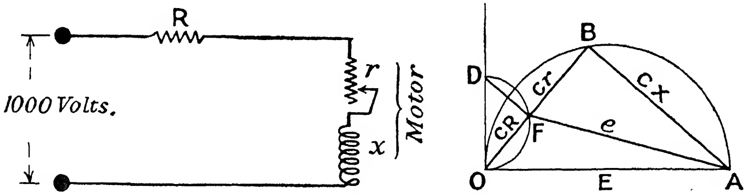
CHAPTER X

EXAMPLES OF THE APPLICATION OF LOCUS DIAGRAMS

EXAMPLE I.—*Transmission of power from source of constant voltage over line of constant resistance.*

Current is supplied from a 3-phase generating station to an induction motor-generator at some distance, over a line having a resistance of 4 ohms per conductor. The generator voltage is 1730, measured between conductors, and the short-circuit current of the induction motor of the motor-generator is 62·5 amperes per phase (star connected), with an angle of lag which may be taken as 90°.

Construct a diagram to show the variation of the motor voltage with the current taken by it, and plot the relation between motor volts and current on a curve.



FIGS. 96 and 97.—Diagram of circuit and circle diagram for motor supplied through line of constant resistance.

In the case of 3-phase transmission problems it is necessary to base all calculations on the current and voltages per phase. In the present case the alternator voltage is

$$\frac{1730}{\sqrt{3}} = 1000 \text{ per phase}$$

We have thus each phase consisting of a constant reactance (of the motor) in series with a constant resistance (of the line) and with a variable "effective resistance." This "effective resistance" is an imaginary resistance in the circuit which would give rise to an energy voltage in the circuit of the same value as the back voltage induced in the motor, as more fully explained on p. 56. Thus, if w represents the watts per phase taken by the motor, the "effective resistance" is given by this equation

$$w = c^2r$$

When r is the "effective resistance,"
c is the curent per phase

The conditions may thus be represented by the circuit diagram (Fig. 96), and the circle diagram (Fig. 97), in which

R represents the constant line resistance = 4 ohms,
r is the variable "effective resistance" of the motor,
X is the constant reactance of the motor.

$$X = \frac{1000}{62 \cdot 5} = 16 \text{ ohms}$$

The present case is an example of Case 4, p. 195, of the previous chapter.

Since X is constant, the length of BA is proportional to C, *i.e.* to the current in the circuit. The length CR, which is to be cut off from OB to represent the voltage spent in the line, is obtained as follows:—

Draw OD vertical and make OD equal on the scale of volts to

$$62 \cdot 5 \times 4 = 250 \text{ volts}$$

i.e. equal to the motor short-circuit current multiplied by the line resistance. On OD describe a semicircle. The length OF cut off from OB by this semicircle will represent the voltage CR spent in the line.

The motor voltage, which is the resultant of the voltages

CX and Cr (see Fig. 97), is the line AF marked e, and its length shows the volts per phase at the motor for any values of the current. The changes of current are shown by changes in length of the line AB.

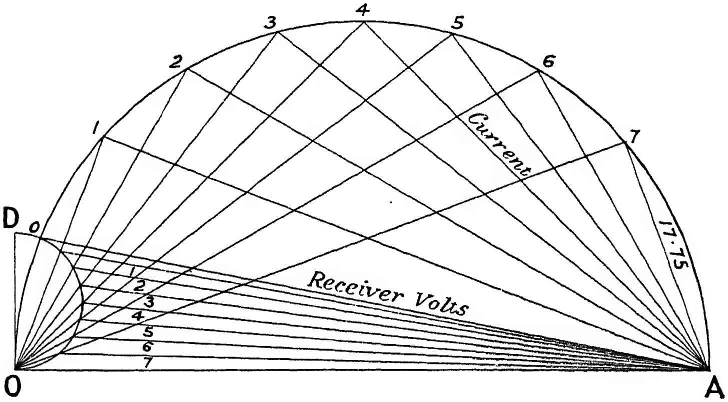


FIG. 98.—Circle diagram for varying load.

From what has been said it will be easy to follow the complete construction of the diagram in Fig. 98, which is the diagram (Fig. 97) drawn to scale for a number of values of r, and the resulting curve in Fig. 99. The curve of

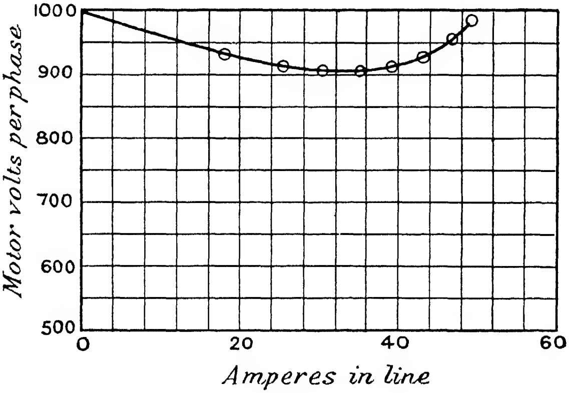


FIG. 99.—Curve of motor voltage with varying load.

voltage drop is interesting as showing that the motor voltage does not drop uniformly as the current increases, but actually shows a rise after a certain value, owing to the large phase difference between the terminal volts and the volts lost in the line when the motor becomes overloaded.

EXAMPLE II.—*Phase Quadrature Device for Induction-type Watt-hour Meters.*—This example is taken as introductory to the examples of transformers and motors which follow, where electrical circuits are linked together magnetically. The selected case shows in a simple manner the similarity existing between circuits connected electrically in parallel and circuits linked together magnetically by being placed on a common magnetic circuit, *i.e.* being connected "magnetically in series."

In a watt-hour meter of the induction type it is necessary, in order to obtain readings which are proportional to the load, that an exact quadrature in phase should be obtained between the magnetic fluxes produced by the shunt and series coils of the meter when working on a non-inductive circuit. When in quadrature of phase, the fields due to the voltage and current coils combine to give a rotating (or "sliding") field which acts upon the armature of the meter and produces the rotation, which in turn actuates the recording mechanism.

The field of the series coil is in phase with the current of the circuit, but the field due to the shunt coil will not be exactly 90° in phase behind the voltage of the circuit, because of the resistance losses in the winding, unless special means are adopted to produce a difference of phase of exactly 90°.

One method of obtaining the required phase difference between the field and the applied voltage of the shunt circuit is to connect a non-inductive resistance *electrically in parallel* with the winding of the electro-magnet, and an inductive resistance in series with the parallel circuits thus formed (see Fig. 100).

The second method, which is "electrically equivalent" to the one just referred to, consists in putting a practically non-inductive circuit *magnetically in series* with the portion of the armature cut by the shunt flux, which is equivalent to connecting them electrically in parallel. An air-gap in a parallel magnetic circuit corresponds to the inductive resistance connected in series with the parallel electrical circuits of the previous arrangement.

Method 1—*Divided Electrical Circuits.*—Let the magnet winding have an impedance z (see Fig. 100), and let it be connected in series with an inductive resistance of impedance z_1, and in parallel with a non-inductive resistance of R ohms.

The conditions to be fulfilled are that the current c in the magnet winding (and consequently the active magnetic flux) shall be in quadrature[1] of phase with the voltage E applied to the terminals of the shunt circuit of the meter, *i.e.* c is to lag 90° behind E.

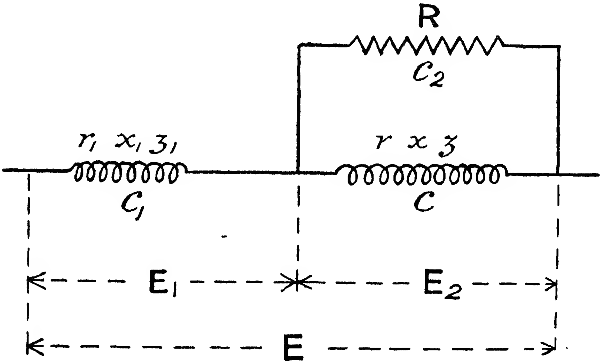


FIG. 100.—Diagram of circuit (Example II.).

Let C be taken as the axis of reference, and let

$$\begin{aligned} E_2 &= C(r - jx) \\ &= rc' - xc'' \end{aligned}$$

be the voltage across the magnet winding.

The current through the resistance R will be

$$C_2 = \frac{E_2}{R} = C \,.\, \frac{r - jx}{R}$$

---

[1] The effect of iron-loss currents is here neglected.

The current C_1 is the sum of these currents, so that

$$C_1 = C + C_2 = C + \frac{1}{R}(r - jx)C$$

The voltage across the series inductive resistance z_1 will be

$$E_1 = C_1(r_1 - jx_1) = C\left(1 + \frac{r - jx}{R}\right)(r_1 - jx_1)$$
$$= C\left\{r_1 + \frac{rr_1}{R} - \frac{xx_1}{R} - j\left(x_1 + \frac{x_1 r}{R} + \frac{xr_1}{R}\right)\right\}$$
$$= c\left(r_1 + \frac{rr_1}{R} - \frac{xx_1}{R}\right)' - c\left(x_1 + \frac{x_1 r}{R} + \frac{xr_1}{R}\right)''$$

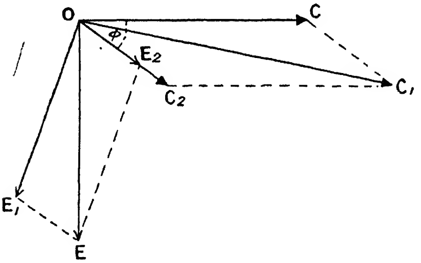


FIG. 101.—Vector diagram, corresponding to Fig. 100.

Thus the total voltage of the circuit

$$E = E_1 + E_2 = c\left(r + r_1 + \frac{rr_1}{R} - \frac{xx_1}{R}\right)' - c\left(x + x_1 + \frac{x_1 r}{R} + \frac{xr_1}{R}\right)''$$

The condition that E shall be perpendicular to C is fulfilled if the parallel component of E is zero, since the axis of reference has been taken along the vector C. Thus, for the required condition of quadrature

$$r + r_1 + \frac{rr_1}{R} - \frac{xx_1}{R} = 0$$

or

$$R = \frac{xx_1 - rr_1}{r + r_1}$$

If the resistance of the series inductive coil can be made negligibly small, so that $r_1 = 0$, the relation would become

$$Rr = xx_1$$

It may be noted in connection with the calculation just given, that only impedances and not admittances have been employed. If the impedance of the parallel circuit R had been a complex number, the use of admittances would have been preferable.

Method 2—*Divided Magnetic Circuit.*—Let us now take the corresponding case, where the resistance R in Fig. 100 is replaced by a practically non-inductive ring placed magnetically in series with the armature disc, and the series inductance x_1 is replaced by a parallel magnetic leakage path. The arrangement is indicated in Fig. 102.

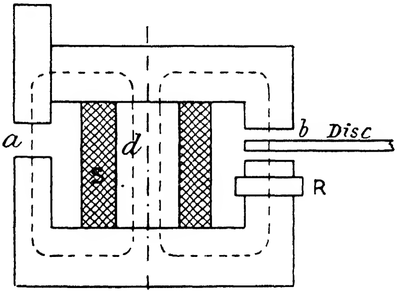


Fig. 102.—Divided magnetic circuit for producing quadrature in phase.

The diagram, which is electrically equivalent to this arrangement, is the one already given in Fig. 100.

The symbols will now have the following meanings:—

E the voltage applied to the magnetizing coil *d*.

E_1 the component of E taken up by the resistance of the coil and the leakage flux through *a*.

E_2 the component of E taken up by the useful flux through *b*, which is also the voltage producing current in the ring R.

x the reactance of coil *d* due to the useful flux.

x_1 the reactance of coil d due to the leakage flux through a.
r_1 the resistance of coil d.
$\mathrm{R} = k^2 \times$ the resistance of the ring R, where k is the

$$\text{ratio } \frac{\text{turns of coil } d}{\text{turns of ring R}}.$$

Further, it is to be remembered that the voltage actually induced in the ring R will be $\frac{e_2}{k}$ volts, and the current actually flowing in the ring R will be kc_2 amps. The values indicated in the diagrams are the "equivalent values" referred to the magnetizing circuit as explained on p. 106.

As before, we shall neglect the iron losses. The effect of the introduction of the leakage path a is thus to add a reactance x_1 to the coil d; the value of r in Fig. 100 is taken to be zero.

The result already obtained from a consideration of the analogous case of a divided electrical circuit, gives as the condition for quadrature in phase of the flux at b and the applied voltage

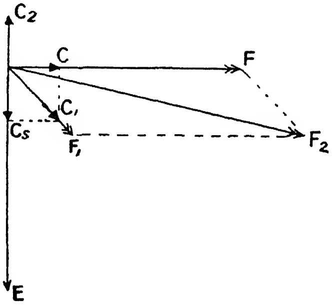


FIG. 103.—Vector diagram corresponding to Fig. 102.

$$\mathrm{R}r_1 = xx_1$$

when r is put equal to zero.

We will proceed to obtain this result by an independent consideration of the electrical circuits which we have stated to be "equivalent" to the circuits illustrated in Fig. 102.

The phase diagram is shown in Fig. 103.

F = useful flux following path through the disc *b*
F_1 = flux following leakage path through air-gap *a*
F_2 = total flux through magnetizing coil *d*
C = component current producing useful flux through *b*
C_2 = current in ring R
C_s = current in the coil *d* balancing C_2

$$= -C_2\frac{t_b}{t} = -\frac{1}{k}C_2$$

where t_b = turns of ring R
t = turns of magnetizing coil *d*
C_1 = total current in coil *d*
E = total applied voltage which is the sum of two voltages, E_1 and E_2, lying along the same vertical line
E_1 = voltage $C_1(r_1 - jx_1)$
E_2 = voltage overcoming induced voltage set up in coil *d* by flux F

The required condition of phase quadrature is that F should be at right angles in phase to E. We have firstly

$$E = E_1 + E_2$$

Now $$E_2 = -jCx$$

since $$r = 0 \text{ (see Fig. 100)}$$

and $$E_1 = C_1(r_1 - jx_1)$$

$$C_1 = C + C_s$$

$$C_s = \frac{E_2}{R} = \frac{-jCx}{R}$$

$$\therefore E_1 = (C + C_s)(r_1 - jx_1)$$

$$= C\left(1 - j\frac{x}{R}\right)(r_1 - jx_1)$$

Taking the axis of reference parallel to the vector F, we may write

$$E_1 = c'\left(1 - j\frac{x}{R}\right)(r_1 - jx_1)$$

$$= c'\left(r_1 - j\frac{xr_1}{R} - jx_1 - \frac{xx_1}{R}\right)$$

and $$E = E_1 + E_2$$

$$= c'\left(r_1 - j\frac{xr_1}{R} - jx_1 - \frac{xx_1}{R} - jx\right)$$

The required condition for quadrature in phase between E and F is that this expression for E shall have no component parallel to F or c'.

Hence $$r_1 - \frac{xx_1}{R} = 0$$

or $$Rr_1 = xx_1$$

which is the result already stated on p. 224.

The example which we have just given was introduced as a simple illustration of the method of treatment of electrical circuits which are not electrically connected, and which yet cannot be considered as independent circuits, because they are linked together by magnetic fluxes. If such a system of magnetically linked circuits is replaced by the "electrically equivalent" circuits, the distribution of currents and voltages may be found as simply as for a network of electrically connected impedances or admittances. Our treatment of the transformer and induction motors is based on this method of regarding the circuits, as will be clearly seen from subsequent Examples (III. and V.), in which the equivalent electrical circuits are drawn as if they formed an electrically continuous system of conductors. It is evident that the vectorial representation of currents and voltages in the circuits (either in symbolic form or by locus diagrams) can thus be made to apply directly to any circuits, whether interconnected electrically or magnetically.

We may state the principle which we have been discussing in the following general form. Electrical circuits which are magnetically in series (*i.e.* which are linked by a common magnetic flux) may be replaced by circuits connected electrically in parallel. Electrical circuits which are magnetically in parallel (*i.e.* which are each linked with one part of a common flux) may be replaced by circuits connected electrically in series.

In making use of this principle, it is necessary to refer all the equivalent circuits to a common standard, as regards number of turns, by multiplication by the ratio of transformation, as explained on p. 105.

Example III.—*The Constant Potential Transformer.*—The general form of the equation for the transformer was given on p. 109 in the form

$$E = C_s(r_1 + k^2r_2 + k^2r + x_1 + k^2x_2 + k^2x) + C_0Z_1$$
$$C_1 = C_s + C_0$$

Since neither the resistance nor the reactance of the load circuit can be taken as constant, no useful application of the circle diagram can be made in this case.

The vector diagram for the transformer has already been given in Chapter V. It is only necessary to give here the equivalent electrical circuit diagram, which is shown in Fig. 104.

It has already been shown (p. 114) that the transformer equation is very approximately

$$E = k^2C_s(Z_2 + Z) + C_sZ_1$$

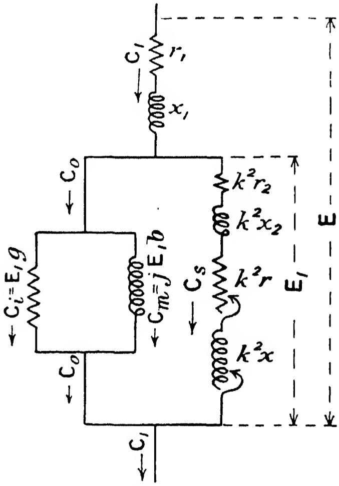


FIG. 104.—Equivalent electrical circuits of transformer.

Now k^2C_sZ is the secondary terminal voltage,

$$\therefore E - C_s(k^2Z_2 + Z_1) = \text{secondary terminal voltage}$$

But $k^2Z_2 + Z_1$ is the equivalent impedance of the transformer windings transferred to the primary circuit (see p. 106), and C_s is nearly equal to C_1.

Adopting this approximation, the equivalent electrical circuit diagram becomes that shown in Fig. 105.

This diagram corresponds to the conditions usually assumed for the purpose of calculating the regulation of a transformer under load from the calculated or test value of the equivalent impedance, and a given load current and power-factor. The usual Kapp diagram for obtaining the regulation at any load and power-factor is based on the assumptions indicated by Fig. 105.

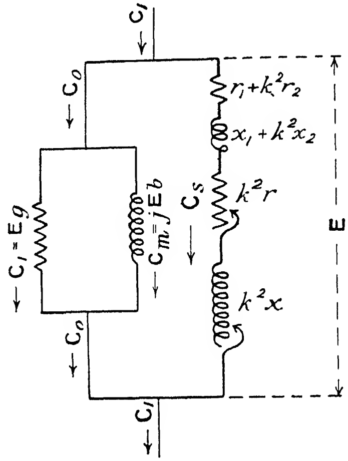


FIG. 105.—Approximate equivalent electrical circuits of transformer.

EXAMPLE IV.—*The Series Alternating Current Motor.*—If we neglect the iron losses, the voltages in the circuit formed by a series motor may be considered to consist simply of an energy voltage proportional to the current overcoming the resistance of the armature and field windings, an idle voltage also proportional to the current overcoming the reactance of the armature and field, and an energy voltage depending on the speed and overcoming the armature rotational voltage. The value of this variable voltage has been given on p. 148 in the form

$$C\frac{x_m}{k}\frac{n}{\sim}$$

Assuming the total applied voltage to be constant, and the variation in reactance due to saturation to be negligible, we have an example of Case 7, p. 203. The annexed diagram (Fig. 106) will be clear from the constructions given on p. 197.

OF = voltage overcoming resistance of field winding
FF_1 = voltage overcoming resistance of armature winding
BG = voltage overcoming reactance of field winding

GA = voltage overcoming reactance of armature due to its self-induction
F_1A = voltage overcoming armature rotational voltage
FG = total voltage across armature brushes

We can render the armature practically non-inductive by adding a neutralizing winding to the motor in such a position as to produce a flux which is equal and opposite to the armature self-induction. The effect of this winding is to reduce the voltage GA to zero, and to bring G to coincide with the point A. Such a change in the conditions will alter the scale to which BA represents the current, so that the same length of BA would represent $\frac{BA}{BG}$ times as much current as before. From this it is evident that the power-factor of the motor for a given current (or speed) is materially increased by the addition of the neutralizing winding.

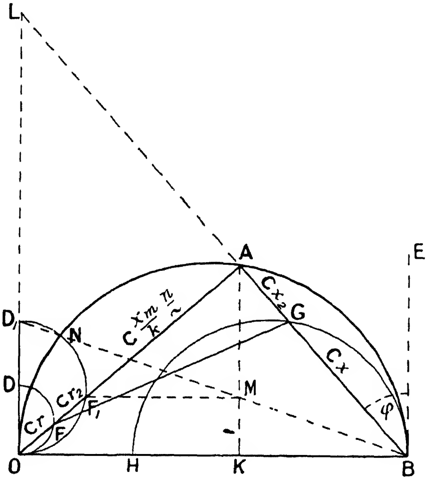


FIG. 106.—Circle diagram for series motor.

It is interesting to observe that the two alternative methods of introducing the neutralizing winding form an excellent illustration of the principle laid down on p. 226, that circuits which are electrically independent, but are joined magnetically in parallel, are equivalent to circuits connected electrically in series. In one form of series motor, the neutralizing winding is an independently short-circuited coil situated on the stator in such a way as to be "magnetically in parallel" with the main field coil, as regards the flux crossing the air-gap. In the alternative arrangement this coil is connected electrically in series with the field winding. Except for secondary considerations of detail, the two arrangements are identical in their effects.

In the complete equation, No. (105), for the series motor given on p. 148,

$$E = C\left(r_2 + r_m + r + \frac{x_m}{k}\frac{n}{\sim}\right) - jC\left(x_2 + x_m + x - \frac{r_m}{k}\frac{n}{\sim}\right)$$

the terms Cr_m and $-jC\frac{r_m}{k}\frac{n}{\sim}$ are due to the iron losses, and are not included in the circle diagram, Fig. 106. Evidently, the first of these terms Cr_m might be included by marking off a small energy voltage along the line OA_1 like the voltages Cr and Cr_2, for which purpose a third circle with vertical diameter might be drawn.

The voltage $-jC\frac{r_m}{k}\frac{n}{\sim}$ is a variable negative reactance voltage, and in order that the line AB may represent the voltage due to a constant reactance (and be proportional to C), it is usual to omit this term. The effect of this term (if considered) would be to tilt the line OB slightly, by bringing B downwards as in Case 5, p. 200. The end of the vector showing the negative reactance voltage would not move on a circle, because the negative reactance varies in value with the speed.

Referring to Fig. 106,

Current is proportional to BA.
BN is the short-circuit current.
Watts input are proportional to AK (see p. 191).

Watts output are proportional to AM (see p. 195).

Copper losses ,, ,, ,, MK.

Power-factor is cosine of angle ABE, *i.e.* is proportional to AO.

Speed is proportional to D_1L, because this length is proportional to F_1A (the induced voltage) divided by BA (proportional to the current and consequently to the flux).

Example V.—*The Polyphase Induction Motor.*—The current and voltage equations for the induction motor as given on pp. 131 and 132 are

$$E = C_s\left(\frac{k^2 \sim}{\sim - n} r_2 + r_1 - jk^2x_2 - jx_1\right) + C_0Z_1$$

$$\begin{aligned} C_1 &= C_s + C_0 \\ &= C_s + E_1Y \end{aligned}$$

The electrical circuit equivalent is shown in Fig. 107, and its connection with the equations just given will be readily followed.

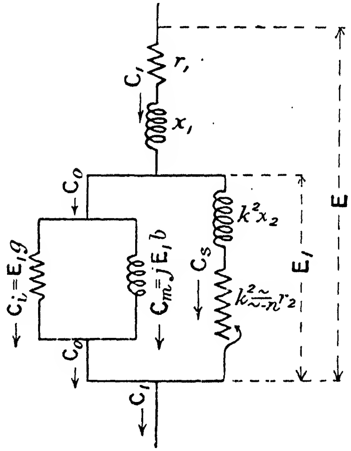


Fig. 107.—Equivalent electrical circuits of the polyphase induction motor.

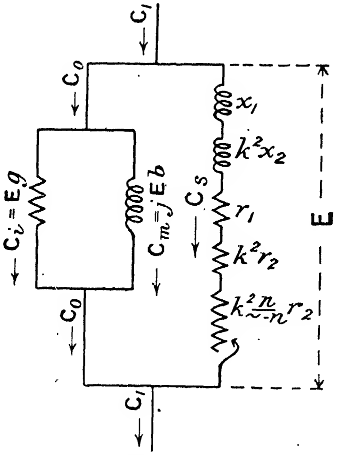


Fig. 108.—Approximate equivalent electrical circuits of polyphase induction motor.

Neglecting the last term, C_0Z_1 (which we have already seen to be permissible in most cases), the expression for E lends itself directly to representation on a constant-voltage circle diagram, in which the reactance is constant, and the current C_s is consequently proportional to the reactance voltage (see Fig. 108).

In order to represent also values of C_1 on a simple circle diagram, it is usual to take C_0 as constant, and to adopt the value $C_0 = EY$, instead of the correct value $C_0 = E_1Y$, which gives the true variation of C_0.

The triangle ABC in Fig. 109 is then the triangle of voltages for the motor, while the triangle OCA is a triangle of currents.

The construction for separating the losses in the constant resistances of the stator and rotor windings is usually carried out as in Case 4, p. 196, the assumption being made that in each case it is the current C_s which flows through these resistances.

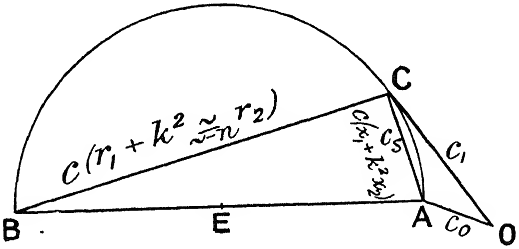


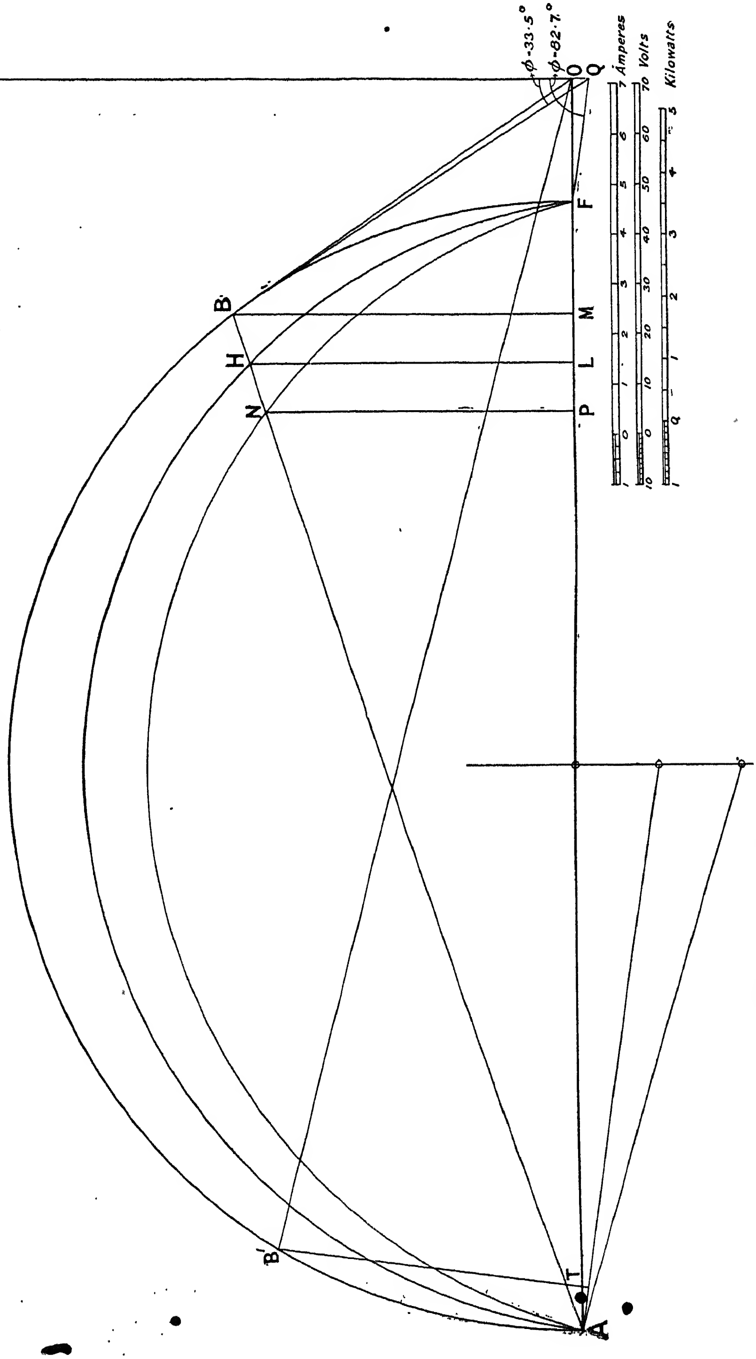
FIG. 109.—Simple circle diagram of polyphase induction motor.

The equivalent electrical circuits as assumed for the usual circle diagram are consequently those shown in Fig. 108.

As an example to illustrate the circle diagram for the Polyphase Induction Motor the case of the machine worked out in Example I., p. 132, is given in Fig. 110 and Fig. 111.

The former shows the regular construction employed by Heyland and others; the latter is more accurate, but much more trouble to construct. The difference between these two brings out the various approximations used in such diagrams, APPROXIMATE CIRCLE DIAGRAM OF INDUCTION MOTOR (3-Phase). (See p. 132.)

Volts 250 ~ 40. H.P. 6.



Poles = 4. Dispersion Coefficient = 0·1

Stator Resistance = 1·3 Ohms.

Equivalent Rotor " = 1·2 "

Stator Reactance } = 5 Ohms each.

Rotor "

[FIG. 110.

and shows to what extent the results agree with those obtained by strict vector algebra.

Referring to Fig. 110, QF is the total magnetizing current which is assumed constant, *i.e.*

$$EY = E_1Y$$

QB is the stator current at the slip specified, and

$$\frac{OF}{OA} = \sigma$$

The resistance drop in the whole machine is represented by BN, BH being that due to the stator coils and HN that due to the rotor. It will be seen that this construction is exactly like that shown in Case 4, and it is therefore obvious that BH and HN are both really proportional to BF, while to be accurate HN should be proportional to BF and BH should be proportional to BQ.

The lengths as measured off from the diagram show the agreement with the values given by the equations on pp. 135 *et seq.*

Referring now to Fig. 111, we have instead of the last case

$$\frac{Oa}{OA} = \sigma$$

Throughout this diagram lines within the large semicircle refer to C_s. Thus, for a rotor current B″O, PC is the rotor resistance drop, AC the stator resistance drop.

The reactance drop in the stator is the intercept D″O, and CD″ is the voltage E_1 to which the magnetizing current is really proportional. So far the diagram is all in accordance with Case 7, p. 204.

Since lines like OB″ represent the rotor current, in order to get the stator current it will be necessary to add a magnetizing current proportional to such lines as CD″. In order to accomplish this the diagram is repeated on a smaller scale at O*a*. Here the semicircle *acb* is proportional to the large semicircle LD″O. The intercepts AC of the circle of stator resistance are represented by intercepts like cd (see Fig. 76), and as a consequence lines like Od are proportional to lines CD'' and parallel to them.

It remains to add a line at right angles to Od proportional to the energy current for the iron loss, and this is done by means of a circle below bda within centre f, so that the intercept dK represents this. OK therefore is the final value of the total magnetizing current corresponding to rotor current OB'' and back E.M.F. CD''; KB'' is then the true value of the stator current. This diagram is accurate except for the line AC, which should be proportional to $B''K$.

The point K travels round its circle as B'' moves round the larger circle, and the variation of the magnetizing current from no load to full load is well shown thereby. These corrections, though of no importance in large induction motors, are of very great importance in very small machines, but it is undoubtedly easier to correct by means of the vector equations than by so elaborate a geometrical drawing.

Example VI.—*Single-phase Induction Motor.*—It has been shown by Ferraris[1] that the behaviour of a single-phase induction motor may be studied by imagining the oscillating flux of the stator to be replaced by two oppositely rotating fluxes, each having a constant value equal to one-half the maximum value of the actual alternating stator flux. We may employ this principle in order to adapt the diagram already given in Fig. 110 for a polyphase motor to represent the conditions for a single-phase induction motor.

Let the stator winding AA' (Fig. 112) of the single-phase motor carry a current producing CT ampere-turns, and giving rise to a field for simplicity shown as forming only two poles which have a vertical axis in Fig. 112. Imagine a winding BB' consisting of two equal sections to be added to the stator, and suppose further that the sections of this winding carry equal and opposite currents, C, each equal to that in the actual stator winding. This imaginary winding BB' is supposed to be wound on the stator so as to form a field perpendicular in space to that of the actual winding, and to be supplied with current which is at right angles in ph the actual stator current. The addition of this wi would not in any way alter the actual conditions of wo

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[1] *Electrician*, 1894, p. 110. See also p. 245 of this book.

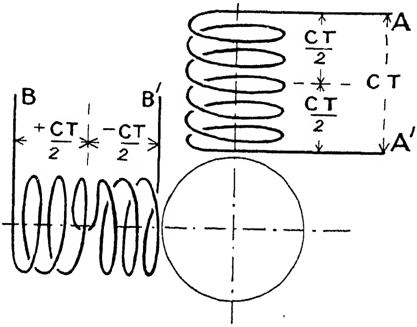

FIG. 112.—Stator of single-phase motor, with imaginary winding added.

since the added winding will carry equal and op ampere-turns in its two sections.

We may now look upon the rotor as being acted up the following stator currents :—

I. $\frac{CT}{2}$ ampere-turns in coil A.
$\frac{CT}{2}$ ampere-turns in coil B perpendicular to in A in both space and time.

II. $\frac{CT}{2}$ ampere-turns in coil A′.
$-\frac{CT}{2}$ ampere-turns in coil B′ perpendicul those in A′ in both space and time.

Now, each of these pairs of windings A, B and A′, fulfil all the conditions for the production of a rotating exactly as in a two-phase motor. But the direct rotation of the field due to coils A, B will be opposite due to coils A′, B′.

The actual conditions are not in any way changed addition of equal and opposite ampere-turns, so th state of things just described corresponds exactly to t the actual motor. We may thus look upon the rotor as being acted upon by two oppositely rotating fields, each due to a two-phase winding of $\frac{T}{2}$ turns per phase. It can be shown that the effect of each of these fields is independent of that of the other, so that we may, for greater clearness, imagine each field to act on a separate rotor.

It is not difficult to see that the currents induced in the rotor by the two fields will be quite distinct when we remember that the frequencies of the currents will have quite different values. The frequency of the currents due to one field will be s cycles per second, while the frequency of the currents due to the other rotating field will be $2 \sim - s$, where s is the positive slip in cycles per second of the rotor behind the field rotating in the same sense as the rotor; $\sim$ is the frequency of supply.

We thus arrive at the result that the single-phase motor is equivalent to two two-phase motors acting on a common shaft, having their stators connected in series, but their rotors electrically independent.

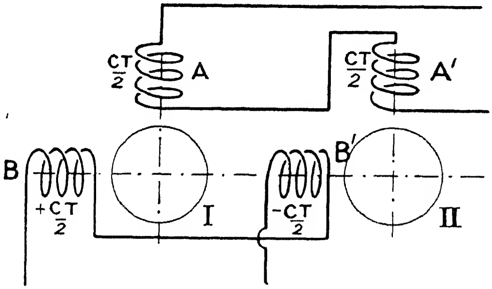

FIG. 113.—Two polyphase motors equivalent to a single-phase motor.

We may thus replace the single-phase motor, having stator coils AA′ and supplied with CT ampere-turns, by two two-phase motors as in Fig. 113, motor I with coils A, B, and motor II with coils A′, B′ on their respective stators, motors I and II being supplied in series and each having a stator current per phase of $\frac{CT}{2}$ ampere-turns.

If these motors are stationary, they will both have a slip of ~ cycles per second, or of 100 per cent., but in opposite directions, since their fields rotate oppositely. Their torques will be equal and opposite, and the resultant torque on their common shaft will consequently be zero. If the shaft has a rotation of n revolutions per second in the sense of the rotation of motor I, the slip of motor I will be $\sim - n$ cycles per second, while the slip of motor II will have become $\sim + n$. For any speed of rotation in either direction, the algebraic sum of the slips of both motors will be $2 \sim$.

The two motors must be considered always to take the same current, since they are connected in series according to our supposition. The voltage measured at their terminals will vary with the relative speeds of the motors. The sum of the two imaginary motor voltages forms the actual voltage of supply of the single-phase motor which has thus a constant value. We have now to determine how this constant voltage is divided between the two polyphase motors.

Let E be the constant total voltage of the single-phase motor, E_1, E_2 the voltages of the two polyphase motors, and C the current per phase taken by these motors.

Further let

$$G_1 = \frac{E_1}{C}$$

and

$$G_2 = \frac{E_2}{C}$$

where G_1, G_2 may be called the total "apparent impedances" of the motors when running (this is, of course, quite a different quantity from the impedance of the motor windings).[1]

We have then the relations

$$E = E_1 + E_2$$

also

$$E_1 = CG_1$$

and

$$E_2 = CG_2$$

Hence

$$\frac{E_1}{E_2} = \frac{G_1}{G_2}$$

---

[1] For approximate value of G_1, G_2, see p. 245.

so long as the motors are considered as being connected in series, so as to take the same current.

The quantities G_1, G_2 will vary with the speed of the motors, but will have a definite value at any given speed. Let us now imagine the two polyphase motors to be connected in *parallel* to the supply circuit (instead of in series) and to be still operating at the previous speed, so that the values of G_1 and G_2 remain the same as before. The currents taken by the motors under the new conditions will be inversely proportional to the "apparent impedance" of each motor. We will suppose the currents to be

$$C_1 = \frac{E}{G_1} \text{ for motor I}$$

$$C_2 = \frac{E}{G_2} \text{ for motor II}$$

whence

$$\frac{C_1}{C_2} = \frac{G_2}{G_1}$$

or, from the previous equation,

$$\frac{E_1}{E_2} = \frac{C_2}{C_1}$$

That is to say, the voltages of the motors when connected *in series* will bear to one another the inverse ratio of the currents which the motors would take if connected *in parallel* to a common supply, while still running at the same speeds.

Thus, if we find (by means of a Heyland diagram or otherwise) the current of one of the two-phase motors when working on the *constant voltage* E, first with a slip $\sim -n$, and then with a slip $\sim +n$, these will be the currents denoted as C_1 and C_2 above. We can from these currents determine the voltages E_1 and E_2 at the terminals of the two motors when connected in series across this voltage, and when their rotors are running at a speed n, corresponding to a speed of n revolutions per second of the single-phase motor.

If we now draw two Heyland diagrams, representing two similar polyphase motors working at voltages E_1 and E_2 and having slips of $\sim - n$ and $\sim + n$, we can obtain the torque, input and output of each motor, by the usual construction as applied to polyphase induction motors. By adding together the values of these quantities obtained for the two motors, with due regard to sign, we shall obtain the torque, input and output of the single-phase motor to which the polyphase motors are equivalent.

We shall show how a complete diagram for the single-phase motor may be drawn, employing for this purpose the construction adopted in the Heyland polyphase motor diagram. The approximations and assumptions of the Heyland construction are made in this diagram, so that we shall obtain a useful approximation, and not a theoretically accurate representation of the performance of the motor.

We may now proceed to the construction of an actual diagram.[1] A Heyland diagram is drawn for the single-phase motor exactly as if it had been provided with a two-phase stator winding having one-half the number of coils per phase of the actual single-phase winding, and working on the full voltage of supply. Such a diagram is shown in the upper part of Fig. 114, where the coefficient of dispersion of the motor $\sigma = \frac{OA}{OB}$. OC' is the short-circuit current, and $C'p$ drawn perpendicular to the radius fB of the torque circle is the slip line, the length pC' representing a slip of 100 per cent. By dividing pC' into ten divisions, and continuing the line beyond C' marked with the same length of subdivision as before, we obtain the means for finding the running conditions for slips greater than 100 per cent., *i.e.* for a backward rotation of the motor. For the purpose of a finer graduation of the slip at high speeds, a second slip line is shown on the right of the diagram, from which slips up to 10 per cent. may be read more exactly.

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[1] See an article on "Single-phase Induction Motor Diagrams," *Mech. Eng.*, Dec. 21, 1907.

For the purposes of explanation of the further method of procedure, we will assume the single-phase motor to be working with a slip of 15 per cent. (choosing a large slip to make the method of construction more distinct).

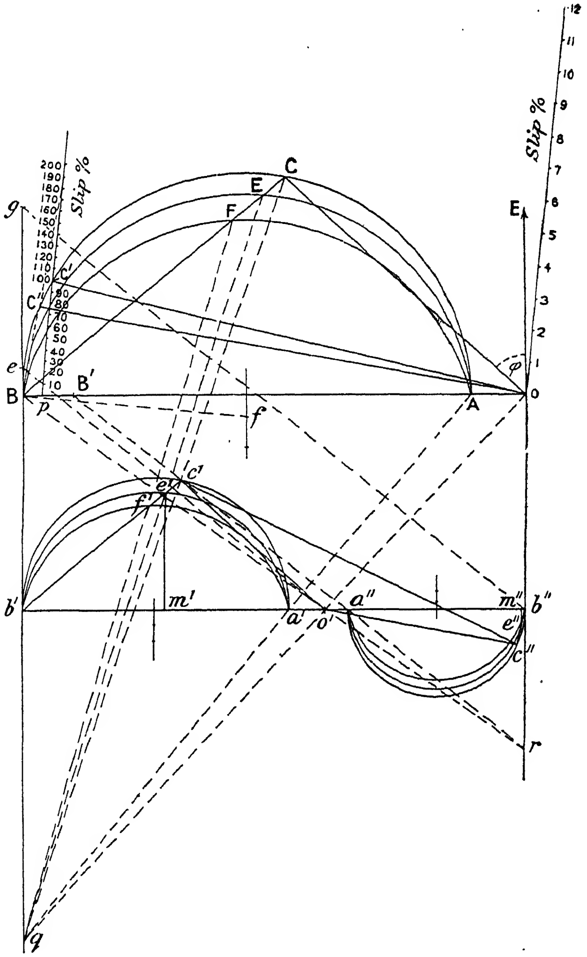

FIG. 114.—Circle diagram for single-phase motor.

The current OC corresponds to a slip of 15 per cent. in Fig. 114, since the line BC is drawn through the 15 per cent. mark on the slip line pC′.

Now, if the polyphase motor I works with a slip of 15 per cent., the motor II will work with a slip of 200 − 15 = 185 per cent. Accordingly the stator current corresponding to a slip of 185 per cent. is determined also, being the line OC''. The point C'' is obtained by joining B to the point 185 per cent. on the slip line, the junction line cutting the input circle at C''.

The currents which would be taken by the polyphase motors I and II when connected severally to the voltage of supply, are represented in phase and magnitude by the lines OC, OC''; in fact, these are the currents C_1 and C_2 previously spoken of. We have now to draw the diagrams for these motors when connected *in series* across the mains, remembering that the voltages which they will receive are inversely proportional to the magnitudes of the currents just found, and that the sum of the voltages is the voltage E represented by the line OB in the upper diagram.

Draw a horizontal line $b''b'$ below the diagram (Fig. 114), making $b''b'$ equal to OB and placing b'' vertically under O. The line $b''b'$ is now to be divided in the ratio $\frac{OC}{OC''}$. This may easily be done by marking off lengths $b'e = OC''$, and $eg = OC$ vertically from b' (in the diagram half these lengths are marked off for the sake of reducing the length of the diagram). Join gb'' and draw eo' parallel to gb'' to cut the line $b''b'$ in o'. The voltage E represented by $b''b'$ is now subdivided so that $b''o'$ represents the voltage of motor II, and $o'b'$ is the voltage of motor I.

On $b''o'$, $o'b'$ we have now to construct the Heyland diagrams which are to represent the performances of the two motors. In order to do this we must first divide each of these lines at points a', a'' in such a way that

$$\frac{o'a'}{o'b'} = \frac{o'a''}{o''b''} = \frac{OA}{OB} = \sigma$$

The point a' is easily found by drawing a line Oo', and producing it to cut the vertical line through Bb' at q. Joining Aq we obtain the required point a'. The point a'' for the motor II is similarly obtained by drawing Bo' to cut the vertical line drawn downwards from O at r, and a second line from a point B′ taken so that BB′ = OB to cut $b'o'$ in a''.

Semicircles may now be drawn on $b'a'$ and on $a''b''$ to represent the locus of the currents taken by the motors I and II. One of these semicircles is inverted for convenience, as in Case 13, p. 208. Drawing $o'c'$ parallel to OC, we have the current of motor I.

Similarly, $o'c''$ drawn parallel to OC″ gives the current of motor II.

From the manner in which the scales of the two polyphase motor diagrams were fixed, the lengths of the two current lines oc' and $o'c''$ must be equal; they are not, however, in phase with one another. This appears at first not to be in accordance with our conception of the motors as being connected in series. The actual conditions governing the production of the two rotating fields are that both fields are obtained from the same source of voltage; and, while both of our imaginary polyphase motors take the same amount of current, the *phase-difference between these currents and their respective voltages* will be dependent on the slip of the motors. The relative phases of the two currents, if E_1 and E_2 were in phase, would be those shown in the diagram, so that the resultant current will be the line $c'c''$ joining the extremities.[1] This is the current actually taken by the stator winding of the single-phase motor.

The torque and output circles may now be put into the smaller diagrams from the larger one. This is done directly in the case of the left-hand diagram by drawing radiating lines through CEF in the large diagram to the point q to cut the line $c'b'$ at points $c'e'f'$, through which the circles must pass, as well as through $b'a'$. The circles in the right-hand diagram (for motor II) are directly derived from those on the left, since lines drawn through o', tangential to the circles of one diagram, will also be tangential to those of the other.

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[1] For further discussion of this point, see p. 246.

The two diagrams which have now been drawn on $b'b''$ represent completely the performance of the single-phase motor when running at a certain speed, or load. Thus, for example, the torque of motor I is given by the line $e'm'$, and that of motor II by $e''m''$. The torque of the single-phase motor is the difference between these, since motor II always exercises a negative torque, acting, in fact, as a generator. Similarly, the resultant output is given by the difference of the ordinates of the output circles of the two motors. The resultant rotor current of the single-phase motor is the resultant of the lines $a'c'$ and $a''c''$ representing the respective rotor currents of the two motors.

The diagram shown in Fig. 114 suffers from the disadvantage of showing the conditions for one value of the slip only. In many practical cases this is sufficient, since the diagram is of most importance when showing the conditions of working at full load. It is easy to see that a decrease in the slip of the single-phase motor will correspond to a movement upwards of the line $b'b''$, the points a' and o' travelling along the polar lines qA and qO, producing an increase in the dimensions of the left-hand diagram at the expense of the one on the right. Similarly, the points $c'e'f'$ will follow the polar lines Cq, Eq, and Fq, as the latter change their inclination with the movement of the points CEF towards A. An increased slip will similarly correspond to a movement of the points $a'o'$, etc., towards q. By means of a piece of tracing paper laid over a drawing of the original diagram, it is thus possible to obtain a number of diagrams, corresponding to different loads with considerable rapidity.[1]

Connection of Preceding Diagram with the Vector Equation. It is of interest to show the connection between the diagrams just given for two polyphase motors connected in series as the equivalent of a single-phase motor and the vector equations previously given.

Referring to equation (88), p. 128, we have the E.M.F. induced in the ideal simple rotor expressed as follows :—

---

[1] The diagram just given is a modified form of that given by Heubach (see Der Drehstrommotor, J. Springer, Berlin).

$$E_r = 2\pi \sim F \cos\theta \sin\lambda + 2\pi n F \sin\theta \cos\lambda$$

where λ is the angular displacement of the rotor coil from the axis of one pole.

By expanding this we obtain

$$E_r = -2\pi\frac{F}{2}(\sim - n)\sin(\theta - \lambda) + 2\pi\frac{F}{2}(\sim + n)\sin(\theta + \lambda)$$

Reference to p. 128 will show that the first half of this expression is an E.M.F. having one-half the value of that induced in the case of the polyphase motor. The second part of the expression is the E.M.F. which would be produced in the armature of an alternator having a frequency of $(\sim + n)$ and a field strength of $\frac{F}{2}$. This E.M.F. is the same as would be produced in the rotor of a polyphase motor running at a speed of n revolutions per second in a direction contrary to that of the rotating field, and having a field strength equal to one-half that of the actual single-phase motor.

Thus we see that the E.M.F. set up in the rotor of the single-phase motor, as expressed by the vector equation, is the vector sum of the E.M.F.'s which would be developed in two similar polyphase motors, one of which is imagined as running with the rotating field, and the other against it. Each motor must be conceived, further, to have half the stator ampere-turns per phase of the original single-phase motor, and both motors must be looked upon as being connected in series, or as taking equal currents.

Again, if the principles enunciated by Ferraris and used for the preceding diagram are correct, then by adding together the vector equations for two polyphase motors, one running with a slip of $\sim + n$, and the other with a slip of $\sim - n$, we should get a close approximation to equation (99), p. 140, which has been deduced from other considerations.

Let us take, then, two such polyphase motors carrying equal currents, and let their E.M.F.'s be (as on p. 238) E_1 and E_2.

Then by pp. 131 and 132 and equation (93), neglecting the small voltage C_0Z_1

$$E_1 = k^2\frac{C_s}{2}\left(\frac{\sim}{\sim - n}r_2 - jx_2\right) + \frac{C_s}{2}Z_1$$

and

$$E_2 = k^2\frac{C_s}{2}\left(\frac{\sim}{\sim + n}r_2 - jx_2\right) + \frac{C_s}{2}Z_1$$

For we have shown that the rotor currents for the two motors are equal, and each is equal to half the actual rotor current.

Adding the above equations

$$E_1 + E_2 = k^2C_s\left(\frac{\sim^2}{\sim^2 - n^2}r_2 - jx_2\right) + C_sZ_1$$

But by hypothesis

$$E_1 + E_2 = E$$

Hence the above equation is identical with (99), except that the left-hand part should read

$$E\left\{1 - \frac{\sim + n}{\sim}(g + jb)(r_1 - jx_1)\right\}$$

instead of E, *i.e.* the diagram (Fig. 114) being a Heyland diagram, neglects the small voltage drop due to the magnetizing current, which we have already shown to be the case with all Heyland diagrams (*cf.* Figs. 107 and 108).

This proof of the identity of the diagram (Fig. 114) with the results given by the vector equation, throws an interesting sidelight on the meaning of some of the expressions which we used to develop the diagram. For instance, it is evident that G_1 and G_2 are very nearly expressed as follows :—

$$G_1 = \frac{k^2}{2}\left(\frac{\sim}{\sim - n}r_2 - jx_2\right) + \frac{Z_1}{2}$$

and

$$G_2 = \frac{k^2}{2}\left(\frac{\sim}{\sim + n}r_2 - jx_2\right) + \frac{Z_1}{2}$$

This is not quite true on account of the approximations already referred to as necessary in the case of the simplified diagram.

These values for G_1 and G_2, however, afford a full explanation for the phase difference existing between $o'c'$ and $o'c''$ already referred to on p. 243. The fact is (as explained for similar lines in connection with Fig. 109) that they do not represent actually currents, but voltages necessary to produce those currents flowing in the impedances G_1 and G_2 when the applied voltages are E_1 and E_2 respectively. Since the components of these impedances involving r and x respectively are not equal, it follows that though the currents flowing through them may be in phase the voltages necessary to produce these currents cannot be represented as lying along one line; indeed, the problem really resolves itself into finding the proper voltage for driving the current C_s through the joint impedance

$$k^2\left(\frac{\sim^2}{\sim^2 - n^2} r_2 - jx_2\right) + Z_1$$

This voltage is represented by $c'c''$, but since at constant speed the above impedance is constant, the same line will also represent the current.

A close agreement between the vector equations and the diagrams cannot be looked for, since the same approximations and assumptions are made as in the simplified Heyland diagram already described. The general method of treatment is, however, shown to be correct.

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